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A New Product for Soft Sets with Its Decision-Making: Soft Star-Product

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Abstract

Soft sets provide a robust mathematical framework for addressing uncertainty, offering innovative solutions for problems involving parametric data. Soft set operations are core concepts within soft set theory. In this paper, we introduce a novel soft product for soft sets, termed the soft star-product, along with its complete algebraic properties concerning various types of soft equalities and subsets. By analyzing the distributions of the soft star product over different soft set operations, we also explore the relationships between this product and other soft set operations. The paper concludes with an example that illustrates the method's effectiveness in various applications, utilizing the int-uni operator and the int-uni decision function within the soft star-product for the int-uni decision-making method, which identifies an optimal set of elements from available alternatives. This work is a valuable contribution to the soft set literature, as the theoretical foundation of soft computing methods is based on rigorous mathematical principles.

Keywords: Soft set, Soft star-product, Soft subset, Soft equal relations, Decision-making.

1 | Introduction

Modern set theory, formulated by George Cantor, forms the foundation of all mathematics. One challenge linked to the concept of a set is vagueness, as mathematics requires precision in all concepts, including sets. This ambiguity or representation of imperfect knowledge has long been an issue for philosophers, logicians, and mathematicians. Recently, it has also become a pressing concern for computer scientists, particularly in artificial intelligence. Various mathematical tools are available for modeling complex systems, such as

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probability theory, fuzzy set theory [1], and interval mathematics, yet each has inherent limitations. Probability theory applies only to stochastically stable systems, while interval mathematics struggles with varying uncertainties, and setting membership values is a known challenge in fuzzy set theory. Moreover, these tools lack parameterization, limiting their effectiveness, especially in complex domains like economics, environmental science, and social sciences. Soft set theory was introduced by Russian researcher Molodtsov [2] in 1999, who proposed it as a fully generic mathematical tool for modeling uncertainty. With no strict constraints on object descriptions, researchers can tailor parameters as needed, greatly simplifying and enhancing the efficiency of decision-making processes, especially when faced with incomplete information. Soft set theory stands out by overcoming the difficulties and offering broader applicability across multidimensional fields. A soft set offers an approximate description of an object, consisting of two parts, a predicate and an approximate value set. In classical mathematics, exact solutions to models are necessary, but for complex models without exact solutions, approximate methods are used.

In contrast, in soft set theory, since the initial description of an object is inherently approximate, there is no need for an exact solution concept. Molodtsov [2] demonstrated the versatility of soft set theory by applying it successfully in diverse areas, including function smoothness, game theory, operations research, and Riemann integration. Several scholars [3–9] developed early soft set-based decision-making methods after Maji et al. [10] first applied soft set theory to a decision-making problem. Çağman and Enginoğlu [11] introduced the "uni-int decision-making" method, a prominent soft set-based approach, and later presented soft matrix-based decision methods for the OR, AND, AND-NOT, and OR-NOT operations [12]. These methods have since proven effective in handling uncertainty and other real-world problems, leading to extensive applications of soft set theory in decision-making [13–22].

In recent years, significant advancements have been made in the foundational aspects of soft set theory. Maji et al. [25] offered a comprehensive theoretical framework that included soft subsets, soft set equality, and soft set operations such as union, intersection, and AND/OR products. Pei and Miao [26] further refined these concepts, redefining intersection and subset relations and examining connections to information systems. Ali et al. [27] introduced additional operations, including restricted union, restricted intersection, restricted difference, and extended intersection. Subsequent research [28–37] examined the algebraic structure of soft sets, proposed improvements, and addressed conceptual inconsistencies in previous soft set studies. In recent contributions, Eren and Çalişici [41] defined a novel type of difference operation for soft sets, while Stojanovic [42] explored the extended symmetric difference of soft sets. Other innovative soft set operations have since been developed and analyzed [43–48]. Soft equality relations and subsets are core concepts in soft set theory. Maji et al. [25] initially proposed a notion of soft subsets, later extended by Pei and Miao [26] and Feng et al. [29]. Qin and Hong [49] introduced two new types of congruence relations and soft equal relations on soft sets. Jun and Yang [50] further expanded soft equal relations, utilizing a broader range of soft subsets to modify Maji's soft distributive laws. This work introduced J-soft equal relations for consistency. Inspired by these developments, Liu et al. [51] proposed soft L-subsets and soft L-equal relations, noting that not all soft equalities conform to distributive norms.

Building on prior work, Feng et al. [52] expanded on the types of soft subsets and examined the algebraic properties of soft product operations, covering laws of distribution, commutativity, and association, among other characteristics. Their work also explored soft products, such as AND and OR products, using soft L-subsets, examining these operations under J-equality and L-equality. They further demonstrated that soft L-equal relations align with commutative semigroup structures. For additional insights on soft equal relations, see [53–57]. Çağman and Enginoğlu [11] enhanced Molodtsov's original concept of soft sets and introduced several products in soft set theory, including *uni-int* decision functions, AND-products, OR-products, AND-NOT-products, and OR-NOT-products. They developed a structured decision-making approach to select optimal options from alternatives using these products, providing a practical example of how this method can address uncertainty. Sezgin et al. [58] conducted a detailed analysis of the AND-product, examining its algebraic properties (e.g., idempotent, commutative, and associative laws) and comparing these with properties related to different soft equalities like soft F, M, L, and J equalities. The study demonstrated that

the collection of all soft sets over the universe forms a commutative hemiring with identity under soft L-equality when the restricted or extended union is used in conjunction with the AND-product. Furthermore, it was established that the same structure holds when the restricted or extended symmetric difference is combined with the AND-product, resulting in another commutative hemiring with identity under the framework of soft L-equality.

In this paper, we introduce a new product operation in soft set theory called the soft star-product. We provide an example of this operation, discussing its algebraic properties in relation to different types of soft subsets and equalities, including M-subset/equality, F-subset/equality, L-subset/equality, and J-subset/equality. The distributional properties of this product over other soft set operations are also examined. Finally, we apply the *int-uni* decision-making method for soft star-product to select optimal options in a decision-making scenario, with an example illustrating its effectiveness. This research contributes to the literature on soft sets by advancing theoretical foundations essential for soft computing applications. The paper is organized as follows: Section 2 provides an overview of key soft set theory concepts. In Section 3, we introduce the soft star-product and discuss its algebraic properties with respect to various soft equalities and subsets. Section 4 explores the application of soft star-product with int-uni decision operators in decision-making. Concluding remarks are presented in the final section.

2 | Preliminaries

Definition 1 ([1]). Let U be the universal set, E be the parameter, and $P(U)$ be the power set of U and $\mathcal{K} \subseteq E$. A pair $(\mathfrak{J}, \mathcal{K})$ is called a soft set over U where $\mathfrak{J}$ is a set-valued function such that $\mathfrak{J}: \mathcal{K} \rightarrow P(U)$.

Although Çağman and Enginoğlu [11] modified Molodstov's concept of soft sets, we continue to use the original definition of soft sets in our work. Throughout this paper, the collection of all the soft sets defined over U is designated as $S_E(U)$. Let $\mathcal{K}$ be a fixed subset of E and $S_{\mathcal{K}}(U)$ be the collection of all those soft sets over U with the fixed parameter set $\mathcal{K}$. That is, while in the set $S_{\mathcal{K}}(U)$, there are only soft sets whose parameter sets are $\mathcal{K}$; in the set $S_E(U)$, there are soft sets whose parameter sets may be any set. From now on, while soft set will be designated by SS and parameter set by PS, soft sets will be designated by SSs and parameter sets by PSs for the sake of ease.

Definition 2 ([27]). Let $(\mathfrak{J}, \mathcal{K})$ be an SS over U. $(\mathfrak{J}, \mathcal{K})$ is called a relative null SS (with respect to the PS $\mathcal{K}$), denoted by $\emptyset_{\mathcal{K}}$, if $\mathfrak{J}(\hbar) = \emptyset$ for all $\hbar \in \mathcal{K}$ and $(\mathfrak{J}, \mathcal{K})$ is called a relative whole SS (with respect to the PS $\mathcal{K}$), denoted by $U_{\mathcal{K}}$ if $\mathfrak{J}(\hbar) = U$ for all $\hbar \in \mathcal{K}$. The relative whole SS U_E with respect to the universe set of parameters E is called the absolute SS over U.

The empty SS over U is the unique SS over U with an empty PS, represented by $\emptyset_{\emptyset}$. Note $\emptyset_{\emptyset}$ and $\emptyset_{\mathcal{M}}$ are different [39]. In the following, we always consider SSs with non-empty PSs in the universe U unless otherwise stated. The concept of soft subset, which we refer to here as soft M-subset to prevent confusion, was initially defined by Maji et al. [25] in the following extremely strict way:

Definition 3 ([25]). Let $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ be two SSs over U. $(\mathfrak{J}, \mathcal{K})$ is called a soft M-subset of $(\mathfrak{S}, \mathcal{Z})$ denoted by $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_M (\mathfrak{S}, \mathcal{Z})$ if $\mathcal{K} \subseteq \mathcal{Z}$ and $\mathfrak{J}(\hbar) = \mathfrak{S}(\hbar)$ for all $\hbar \in \mathcal{K}$. Two SSs $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ are said to be soft M-equal, denoted by $(\mathfrak{J}, \mathcal{K}) =_M (\mathfrak{S}, \mathcal{Z})$ if $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_M (\mathfrak{S}, \mathcal{Z})$ and $(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_M (\mathfrak{J}, \mathcal{K})$.

Definition 4 ([26]). Let $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ be two SSs over U. $(\mathfrak{J}, \mathcal{K})$ is called a soft F-subset of $(\mathfrak{S}, \mathcal{Z})$ denoted by $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_F (\mathfrak{S}, \mathcal{Z})$ if $\mathcal{K} \subseteq \mathcal{Z}$ and $\mathfrak{J}(\hbar) \subseteq \mathfrak{S}(\hbar)$ for all $\hbar \in \mathcal{K}$. Two SSs $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ are said to be soft F-equal, denoted by $(\mathfrak{J}, \mathcal{K}) =_F (\mathfrak{S}, \mathcal{Z})$ if $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_F (\mathfrak{S}, \mathcal{Z})$ and $(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_F (\mathfrak{J}, \mathcal{K})$.

It is important to note that the definitions of soft F-subset and soft F-equal were originally introduced by Pei and Miao in [26]. However, some papers on soft subsets and soft equalities mistakenly attribute these definitions to Feng et al. [29]. Consequently, the letter "F" is used to reference this connection.

In [52], it was shown that the soft equality relations =M and =F are equivalent. In other words, $((\mho, \mathcal{M}) =_M (\mathfrak{F}, \mathcal{D})$ if and only if $(\mho, \mathcal{M}) =_F (\mathfrak{F}, \mathcal{D})$. Since they have the same set of parameters and

approximation function, two SSs that satisfy this equivalence are actually identical [52], meaning that $(\mho,\mathcal{M}) =_{\mathrm{M}} (\mathfrak{F},\mathcal{D})$ implies $(\mho,\mathcal{M}) = (\mathfrak{F},\mathcal{D})$.

Jun and Yang [51] expanded the concepts of F-soft subsets and soft F-equal relations by relaxing the restrictions on Parameter Sets (PSs). Although they referred to these as the generalized soft subset and generalized soft equal relation in [51], we refer to them as soft J-subsets and soft J-equal relations, taking the initial letter of Jun.

Definition 5 ([51]). Let $(\mathfrak{I},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be two SSs over U. $(\mathfrak{I},\mathcal{K})$ is called a soft J-subset of $(\mathfrak{S},\mathcal{Z})$ denoted by $(\mathfrak{I},\mathcal{K}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{S},\mathcal{Z})$ if for all $\mathscr{k} \in \mathcal{K}$, there exists $z \in \mathcal{Z}$ such that $\mathfrak{I}(\mathscr{k}) \subseteq \mathfrak{S}(z)$. Two SSs $(\mathfrak{I},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ are said to be soft J-equal, denoted by $(\mathfrak{I},\mathcal{K}) =_{\mathrm{J}} (\mathfrak{S},\mathcal{Z})$ if $(\mathfrak{I},\mathcal{K}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{S},\mathcal{Z}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{I},\mathcal{K})$.

In [52] and [53], it was demonstrated that $(\mho,\mathcal{M}) \tilde{\subseteq}_{\mathrm{M}} (\mathfrak{F},\mathcal{D}) \Rightarrow (\mho,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{D}) \Rightarrow (\mho,\mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F},\mathcal{D})$, but the converse may not be true.

Liu, Feng, and Jun [52] introduced a new type of soft subset, referred to as soft L-subsets and soft L-equality, which generalizes both soft M-subsets and ontology-based soft subsets. This new concept was inspired by the ideas of soft J-subsets [51] and ontology-based soft subsets [30].

Definition 6 ([52]). Let $(\mathfrak{I},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be two SSs over U. $(\mathfrak{I},\mathcal{K})$ is called a soft L-subset of $(\mathfrak{S},\mathcal{Z})$ denoted by $(\mathfrak{I},\mathcal{K}) \tilde{\subseteq}_{\mathrm{L}} (\mathfrak{S},\mathcal{Z})$ if for all $\mathscr{k} \in \mathcal{K}$, there exists $z \in \mathcal{Z}$ such that $\mathfrak{I}(\mathscr{k}) = \mathfrak{S}(z)$. Two SSs $(\mathfrak{I},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ are said to be soft J-equal, denoted by $(\mathfrak{I},\mathcal{K}) =_{\mathrm{L}} (\mathfrak{S},\mathcal{Z})$ if $(\mathfrak{I},\mathcal{K}) \tilde{\subseteq}_{\mathrm{L}} (\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{S},\mathcal{Z}) \tilde{\subseteq}_{\mathrm{L}} (\mathfrak{I},\mathcal{K})$.

Concerning the relationships among various types of soft subsets and soft equalities $(\mathfrak{I},\mathcal{K}) \tilde{\subseteq}_{\mathrm{M}} (\mathfrak{S},\mathcal{Z}) \Rightarrow (\mathfrak{I},\mathcal{K}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{S},\mathcal{Z}) \Rightarrow (\mathfrak{I},\mathcal{K}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{I},\mathcal{K}) =_{\mathrm{M}} (\mathfrak{S},\mathcal{Z}) \Rightarrow (\mathfrak{I},\mathcal{K}) =_{\mathrm{L}} (\mathfrak{S},\mathcal{Z}) \Rightarrow (\mathfrak{I},\mathcal{K}) =_{\mathrm{J}} (\mathfrak{S},\mathcal{Z})$ [52]. However, the converses may not be true. Also, it is well-known that $(\mathfrak{I},\mathcal{K}) =_{\mathrm{M}} (\mathfrak{S},\mathcal{Z})$ if and only if $(\mathfrak{I},\mathcal{K}) =_{\mathrm{F}} (\mathfrak{S},\mathcal{Z})$.

We can thus conclude that soft M-equality (and therefore soft F-equality) represents the strictest form of soft equality, while soft J-equality is the weakest. Positioned between these two is the concept of soft L-equality [52].

For further information on soft F-equality, soft M-equality, soft J-equality, soft L-equality, and other definitions of soft subsets and soft equal relations in the literature, please refer to [49–57].

Definition 7 ([27]). Let $(\mathfrak{I},\mathcal{K})$ be two SS over U. The relative complement of $(\mathfrak{I},\mathcal{K})$, denoted by $(\mathfrak{I},\mathcal{K})^{\mathrm{r}}$, is defined by $(\mathfrak{I},\mathcal{K})^{\mathrm{r}} = (\mathfrak{I}^{\mathrm{r}},\mathcal{K})$, where $\mathfrak{I}^{\mathrm{r}}: \mathcal{K} \to \mathrm{P(U)}$ is a mapping given by $\mathfrak{I}^{\mathrm{r}}(\mathscr{k}) = \mathrm{U}\setminus\mathfrak{I}(\mathscr{k})$ for all $\mathscr{k} \in \mathcal{K}$. From now on, $\mathrm{U}\setminus\mathfrak{I}(\mathscr{k})=[\mathfrak{I}(\mathscr{k})]'$ is designated by $\mathfrak{I}'(\mathscr{k})$ for the sake of designation.

Definition 8 ([25]). Let $(\mathfrak{I},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be two SSs over U. The AND-product ($\wedge$-product) of $(\mathfrak{I},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$, denoted by $(\mathfrak{I},\mathcal{K})\wedge(\mathfrak{S},\mathcal{Z})$, is defined by $(\mathfrak{I},\mathcal{K})\wedge(\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K}\mathrm{x}\mathcal{Z})$, where for all $(\mathscr{k},z) \in \mathcal{K}\mathrm{x}\mathcal{Z}$, $\mathfrak{Q}(\mathscr{k},z) = \mathfrak{I}(\mathscr{k}) \cap \mathfrak{S}(z)$.

Definition 9 ([25]). Let $(\mathfrak{I},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be two SSs over U. The OR-product $\vee$ (-product) of $\mathfrak{I},\mathcal{K})$ (and $(\mathfrak{S},\mathcal{Z})$, denoted by $(\mathfrak{I},\mathcal{K}) \vee (\mathfrak{S},\mathcal{Z})$, is defined by $(\mathfrak{I},\mathcal{K}) \vee (\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K}\mathrm{x}\mathcal{Z})$, where for all $(\mathscr{k},z) \in \mathcal{K}\mathrm{x}\mathcal{Z}$, $\mathfrak{Q}(\mathscr{k},z) = \mathfrak{I}(\mathscr{k}) \cup \mathfrak{S}(z)$.

Çağman [59] introduced the concepts of inclusive complement and exclusive complement as novel ideas in set theory, examining the relationships between them through comparison. These new concepts were also applied to group theory in [59]. Sezgin et al. [60] introduced some new complements, investigated the relations between them, and applied them to group theory as well.

Definition 10 ([60]). Let A and B be two subsets of the universe. Then, A star B is defined by $\mathrm{A} * \mathrm{B} := \mathrm{A}' \cup \mathrm{B}'$ Subsequently, the star operation was applied to SS theory to introduce new SS operations [48], [61–63]. Let $\odot$ represent set operations such as $\cap$, $\cup$,$\setminus$,$\triangle$. The following definitions are provided for restricted, extended, and soft binary piecewise operations.

Definition 11 ([27]). Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. The restricted $\odot$ operation of $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$, denoted by $(\mathfrak{J},\mathcal{K}) \odot_{\mathrm{R}} (\mathfrak{S},\mathcal{Z})$ is defined by $(\mathfrak{J},\mathcal{K}) \odot_{\mathrm{R}} (\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{C})$, where $\mathcal{C} = \mathcal{K} \cap \mathcal{Z}$ and if $\mathcal{C} \neq \emptyset$, then for all $c \in \mathcal{C}$, $\mathfrak{Q}(c) = \mathfrak{J}(c) \odot \mathfrak{S}(c)$; if $\mathcal{C} = \emptyset$, then $(\mathfrak{J},\mathcal{K}) \odot_{\mathrm{R}} (\mathfrak{S},\mathcal{Z}) = \emptyset_{\emptyset}$.

Definition 12 ([27], [42], [61]). Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. The extended $\odot$ operation of $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$, denoted by $(\mathfrak{J},\mathcal{K}) \odot_{\varepsilon} (\mathfrak{S},\mathcal{Z})$ is defined by $(\mathfrak{J},\mathcal{K}) \odot_{\varepsilon} (\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{C})$, where $\mathcal{C} = \mathcal{K} \cup \mathcal{Z}$ and for all $c \in \mathcal{C}$,

$$\mathfrak{Q}(c) = \begin{cases} \mathfrak{J}(c), & c \in \mathcal{K}\setminus \mathcal{Z}, \\ \mathfrak{S}(c), & c \in \mathcal{Z}\setminus\mathcal{K}, \\ \mathfrak{J}(c) \odot \mathfrak{S}(c), & c \in \mathcal{K} \cap \mathcal{Z}. \end{cases} \tag{1}$$

Definition 13 ([43], [63]). Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. The soft binary piecewise $\odot$ operation of $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$, denoted by $(\mathfrak{J},\mathcal{K}) \widetilde{\odot} (\mathfrak{S},\mathcal{Z})$ is defined by $(\mathfrak{J},\mathcal{K}) \widetilde{\odot} (\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K})$, where for all $c \in \mathcal{K}$,

$$\mathfrak{Q}(c) = \begin{cases} \mathfrak{J}(c), & c \in \mathcal{K}\setminus \mathcal{Z}, \\ \mathfrak{J}(c) \odot \mathfrak{S}(c), & c \in \mathcal{K} \cap \mathcal{Z}. \end{cases} \tag{1}$$

For more about the soft algebraic structures of SSs, we refer to [64–89].

3 | Soft Star-Product and Its Algebraic Properties

We propose the soft star-product, a novel product for SSs, in this part. We provide an example and analyze its algebraic characteristics in depth with respect to specific kinds of soft equalities and soft subsets.

Definition 14. Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. The soft star-product of $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$, denoted by $(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{S},\mathcal{Z})$, is defined by $(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{S},\mathcal{Z})= (\mathfrak{Q},\mathcal{K}\mathrm{x}\mathcal{Z})$, where for all $(\hbar,z)\in \mathcal{K}\mathrm{x}\mathcal{Z}$, $\mathfrak{Q}(\hbar,z) = \mathfrak{J}(\hbar) * \mathfrak{S}(z)$.

Here, $\mathfrak{J}(\hbar) * \mathfrak{S}(z) = \mathfrak{J}'(\hbar)\cup \mathfrak{S}'(z)$.

Example 1. Let $\mathrm{E} = \{\ell_1, \ell_2, \ell_3, \ell_4\}$ be the PS, $\mathcal{K} = \{\ell_2, \ell_3\}$, and $\mathcal{Z} = \{\ell_2, \ell_4\}$ be the subsets of E, $\mathrm{U} = \{\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3, \mathfrak{f}_4, \mathfrak{f}_5\}$ be the universal set, $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U such that

$(\mathfrak{J},\mathcal{K}) = \{(\ell_2,\{\mathfrak{f}_1,\mathfrak{f}_2,\mathfrak{f}_3,\mathfrak{f}_4,\mathfrak{f}_5\}),(\ell_3,\{\mathfrak{f}_3,\mathfrak{f}_5\})\}$, $(\mathfrak{S},\mathcal{Z}) = \{(\ell_2,\{\mathfrak{f}_1,\mathfrak{f}_2,\mathfrak{f}_3\})(\ell_4,\{\mathfrak{f}_2,\mathfrak{f}_3,\mathfrak{f}_4\})\}$.

Let $(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K}\mathrm{x}\mathcal{Z})$. Then

$$(\mathfrak{Q},\mathcal{K}\mathrm{x}\mathcal{Z}) = \Big\{\big((\ell_2,\ell_2),\{\mathfrak{f}_4,\mathfrak{f}_5\}\big),\big((\ell_2,\ell_4),\{\mathfrak{f}_1,\mathfrak{f}_5\}\big),\big((\ell_3,\ell_2),\{\mathfrak{f}_1,\mathfrak{f}_2,\mathfrak{f}_4,\mathfrak{f}_5\}\big),\big((\ell_3,\ell_4),\{\mathfrak{f}_1,\mathfrak{f}_2,\mathfrak{f}_4,\mathfrak{f}_5\}\big)\Big\}$$

Since it is more practical than writing in the list method style, the table method can be applied here:

$(\mathfrak{J},\mathcal{K})\ \Lambda_*\ (\mathfrak{S},\mathcal{Z})$	ℓ_2	ℓ_4
ℓ_2	$\{\mathfrak{f}_4,\mathfrak{f}_5\}$	$\{\mathfrak{f}_1,\mathfrak{f}_5\}$
ℓ_3	$\{\mathfrak{f}_1,\mathfrak{f}_2,\mathfrak{f}_4,\mathfrak{f}_5\}$	$\{\mathfrak{f}_1,\mathfrak{f}_2,\mathfrak{f}_4,\mathfrak{f}_5\}$

Proposition 1. V_*-product is closed in $\mathrm{S_E(U)}$.

Proof: it is clear that V_*-product in a binary operation in $\mathrm{S_E(U)}$. In fact, let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. Then,

$$\Lambda_*: \mathrm{S_E(U)} \times \mathrm{S_E(U)} \to \mathrm{S_E(U)},\big((\mathfrak{J},\mathcal{K}),(\mathfrak{S},\mathcal{Z})\big) \to (\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K}\mathrm{x}\mathcal{Z}) = (\mathfrak{Q},\mathcal{C}).$$

That is, $(\mathfrak{Q},\mathcal{C})$ is an SS over U since the set $\mathrm{S_E(U)}$ contains all the SS over U. Here, note that the set $\mathrm{S}_{\mathcal{K}}(\mathrm{U})$ is not closed under V_*-product, since if $(\mathfrak{J},\mathcal{K}),(\mathfrak{S},\mathcal{K})$ are the elements of $\mathrm{S}_{\mathcal{K}}(\mathrm{U})$, $(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{S},\mathcal{K})$ is an element of $\mathrm{S}_{\mathcal{K}\mathrm{x}\mathcal{K}}(\mathrm{U})$, not $\mathrm{S}_{\mathcal{K}}(\mathrm{U})$.

Proposition 2. Let $(\mathfrak{J},\mathcal{K})$, $(\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{Q},\mathcal{C})$ be SSs over U. Then,

$$(\mathfrak{J},\mathcal{K})\mathrm{V}_*[(\mathfrak{S},\mathcal{Z})\mathrm{V}_*(\mathfrak{Q},\mathcal{C})] \neq_{\mathrm{M}} [(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{S},\mathcal{Z})]\mathrm{V}_*(\mathfrak{Q},\mathcal{C}).$$

That is, V_*-product is not associative in $\mathrm{S_E(U)}$.

Proof: we provided an example to show that V_*-product is not associative in $\mathrm{S_E(U)}$. Let $\mathrm{E} = \{\ell_1, \ell_2, \ell_3, \ell_4\}$ be PS, $\mathcal{K} = \{\ell_2, \ell_3\}$, $\mathcal{Z} = \{\ell_1\}$, and $\mathcal{C} = \{\ell_4\}$ be the subsets of E, $\mathrm{U} = \{\not{f}_1, \not{f}_2, \not{f}_3, \not{f}_4, \not{f}_5\}$ be the universal set, $(\mathfrak{I}, \mathcal{K})$, $(\mathfrak{S}, \mathcal{Z})$ and $(\mathfrak{Q}, \mathcal{C})$ be SSs over U such that $(\mathfrak{I}, \mathcal{K}) = \{(\ell_2, \{\not{f}_3, \not{f}_4\}), (\ell_3, \{\not{f}_1\})\}$, $(\mathfrak{S}, \mathcal{Z}) = \{(\ell_1, \emptyset)\}$ and $(\mathfrak{Q}, \mathcal{C}) = \{(\ell_4, \{\not{f}_1, \not{f}_3, \not{f}_5\})\}$. We show that $(\mathfrak{I}, \mathcal{K})\mathrm{V}_*[(\mathfrak{S}, \mathcal{Z})\mathrm{V}_*(\mathfrak{Q}, \mathcal{C})] \neq_{\mathrm{M}} [(\mathfrak{I}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z})]\mathrm{V}_*(\mathfrak{Q}, \mathcal{C})$.

Let $(\mathfrak{S}, \mathcal{Z})\mathrm{V}_*(\mathfrak{Q}, \mathcal{C}) = (\zeta, \mathcal{Z} \times \mathrm{C})$. Then

$(\zeta, \mathcal{Z} \times \mathrm{C}) = \{((\ell_1, \ell_4), \{\not{f}_1, \not{f}_2, \not{f}_3, \not{f}_4, \not{f}_5\})\}$.

Assume that $(\mathfrak{I}, \mathcal{K})\mathrm{V}_*(\zeta, \mathcal{Z} \times \mathrm{C}) = (\mathfrak{E}, \mathcal{K} \times (\mathcal{Z} \times \mathrm{C}))$. Thus,

$(\mathfrak{E}, \mathcal{K} \times (\mathcal{Z} \times \mathrm{C})) = \{((\ell_2, (\ell_1, \ell_4)), \{\not{f}_1, \not{f}_2, \not{f}_5\}), ((\ell_3, (\ell_1, \ell_4)), (\not{f}_2, \not{f}_3, \not{f}_4, \not{f}_5))\}$.

Let $(\mathfrak{I}, \mathcal{K})\ \mathrm{V}_*\ (\mathfrak{S}, \mathcal{Z}) = (\mathfrak{X}, \mathcal{K} \times \mathcal{Z})$. Thereby,

$(\mathfrak{X}, \mathcal{K} \times \mathcal{Z}) = \{((\ell_2, \ell_1), \{\not{f}_1, \not{f}_2, \not{f}_3, \not{f}_4, \not{f}_5\}), ((\ell_3, \ell_1), \{\not{f}_1, \not{f}_2, \not{f}_3, \not{f}_4, \not{f}_5\})\}$.

Suppose that $(\mathfrak{X}, \mathcal{K} \times \mathcal{Z})\mathrm{V}_*(\mathfrak{Q}, \mathcal{C}) = (\zeta, (\mathcal{K} \times \mathcal{Z}) \times \mathcal{C})$. Hence,

$(\zeta, (\mathcal{K} \times \mathcal{Z}) \times \mathcal{C}) = \{(((\ell_2, \ell_1), \ell_4), \{\not{f}_2, \not{f}_4\}), (((\ell_3, \ell_1), \ell_4), \{\not{f}_2, \not{f}_4\})\}$.

Thus, $(\mathfrak{E}, \mathcal{K} \times (\mathcal{Z} \times \mathrm{C})) \neq_{\mathrm{M}} (\zeta, (\mathcal{K} \times \mathcal{Z}) \times \mathcal{C})$. It is also obvious that $(\mathfrak{E}, \mathcal{K} \times (\mathcal{Z} \times \mathrm{C})) \neq_{\mathrm{L}} (\zeta, (\mathcal{K} \times \mathcal{Z}) \times \mathcal{C})$ and $(\mathfrak{E}, \mathcal{K} \times (\mathcal{Z} \times \mathrm{C})) \neq_{\mathrm{J}} (\zeta, (\mathcal{K} \times \mathcal{Z}) \times \mathcal{C})$.

Proposition 3. Let $(\mathfrak{I}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ be SSs over U. Then, $(\mathfrak{I}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) \neq_{\mathrm{M}} (\mathfrak{S}, \mathcal{Z})\mathrm{V}_*(\mathfrak{I}, \mathcal{K})$. Namely, V_*-product is not commutative in $\mathrm{S_E(U)}$.

Proof: let $(\mathfrak{I}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) = (\mathfrak{Q}, \mathcal{K} \times \mathcal{Z})$ and $(\mathfrak{S}, \mathcal{Z})\mathrm{V}_*(\mathfrak{I}, \mathcal{K}) = (\mathfrak{X}, \mathcal{Z} \times \mathcal{K})$. Since $\mathcal{K} \times \mathcal{Z} \neq \mathcal{Z} \times \mathcal{K}$, the rest of the proof is obvious.

Proposition 4. Let $(\mathfrak{I}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ be SSs over U. Then, $(\mathfrak{I}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) =_{\mathrm{L}} (\mathfrak{S}, \mathcal{Z})\mathrm{V}_*(\mathfrak{I}, \mathcal{K})$. Namely, V_*-product is commutative in $\mathrm{S_E(U)}$ under L-equality.

Proof: Let $(\mathfrak{I}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) = (\mathfrak{Q}, \mathcal{K} \times \mathcal{Z})$ and $(\mathfrak{S}, \mathcal{Z})\mathrm{V}_*(\mathfrak{I}, \mathcal{K}) = (\mathfrak{X}, \mathcal{Z} \times \mathcal{K})$. Thus, for all $(\hbar, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{Q}(\hbar, z) = \mathfrak{I}'(\hbar) \cup \mathfrak{S}'(z)$ and for all $(z, \hbar) \in \mathcal{Z} \times \mathcal{K}$, $\mathfrak{X}(z, \hbar) = \mathfrak{S}'(z) \cup \mathfrak{I}'(\hbar)$. Since for all $(\hbar, z) \in \mathcal{K} \times \mathcal{Z}$, there exists $(z, \hbar) \in \mathcal{Z} \times \mathcal{K}$ such that $\mathfrak{Q}(\hbar, z) = \mathfrak{I}'(\hbar) \cup \mathfrak{S}'(z) = \mathfrak{S}'(z) \cup \mathfrak{I}'(\hbar)$, $(\mathfrak{I}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) \subseteq_{\mathrm{L}} (\mathfrak{S}, \mathcal{Z})\mathrm{V}_*(\mathfrak{I}, \mathcal{K})$. Similarly, since for all $(z, \hbar) \in \mathcal{Z} \times \mathcal{K}$, there exists $(\hbar, z) \in \mathcal{K} \times \mathcal{Z}$ such that $\mathfrak{S}'(z) \cup \mathfrak{I}'(\hbar) = \mathfrak{I}'(\hbar) \cup \mathfrak{S}'(z)$, $(\mathfrak{S}, \mathcal{Z})\mathrm{V}_*(\mathfrak{I}, \mathcal{K}) \subseteq_{\mathrm{L}} (\mathfrak{I}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z})$ is obtained. Thereby, $(\mathfrak{I}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) =_{\mathrm{L}} (\mathfrak{S}, \mathcal{Z})\mathrm{V}_*(\mathfrak{I}, \mathcal{K})$. Moreover, $(\mathfrak{I}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) =_{\mathrm{J}} (\mathfrak{S}, \mathcal{Z})\mathrm{V}_*(\mathfrak{I}, \mathcal{K})$ is satisfied.

Proposition 5. Let $(\mathfrak{I}, \mathcal{K})$ be an SS over U. Then, $(\mathfrak{I}, \mathcal{K})\mathrm{V}_*\emptyset_\emptyset =_{\mathrm{M}} \emptyset_\emptyset\mathrm{V}_*(\mathfrak{I}, \mathcal{K}) =_{\mathrm{M}} \emptyset_\emptyset$. Namely, $\emptyset_\emptyset$-the empty SS-is the absorbing element of V_*-product in $\mathrm{S_E(U)}$.

Proof: let $\emptyset_\emptyset = (\mathfrak{Q}, \emptyset)$ and $(\mathfrak{I}, \mathcal{K})\mathrm{V}_*\emptyset_\emptyset = (\mathfrak{I}, \mathcal{K})\mathrm{V}_*(\mathfrak{Q}, \emptyset) = (\mathfrak{S}, \mathcal{K} \times \emptyset) = (\mathfrak{S}, \emptyset)$. Since $\emptyset_\emptyset$ is the only SS whose PS is $\emptyset$, $(\mathfrak{S}, \emptyset) = \emptyset_\emptyset$ is obtained. Similarly, $\emptyset_\emptyset\mathrm{V}_*(\mathfrak{I}, \mathcal{K}) =_{\mathrm{M}} \emptyset_\emptyset$.

Proposition 6. Let $(\mathfrak{I}, \mathcal{K})$ be an SS over U. Then, $(\mathfrak{I}, \mathcal{K})\mathrm{V}_*\emptyset_{\mathcal{K}} =_{\mathrm{M}} \emptyset_{\mathcal{K}}\mathrm{V}_*(\mathfrak{I}, \mathcal{K}) =_{\mathrm{M}} \mathrm{U}_{\mathcal{K}\times\mathcal{K}}$.

Proof: let $\emptyset_{\mathcal{K}} = (\mathfrak{Q}, \mathcal{K})$. Then, for all $\hbar \in \mathcal{K}$, $\mathfrak{Q}(\hbar) = \emptyset$. Let $(\mathfrak{I}, \mathcal{K})\mathrm{V}_*\emptyset_{\mathcal{K}} = (\mathfrak{I}, \mathcal{K})\mathrm{V}_*(\mathfrak{Q}, \mathcal{K}) = (\mathfrak{S}, \mathcal{K} \times \mathcal{K})$, where for all $(\hbar, z) \in \mathcal{K} \times \mathcal{K}$, $\mathfrak{S}(\hbar, z) = \mathfrak{I}'(\hbar) \cup \mathfrak{Q}'(z) = \mathfrak{I}'(\hbar) \cup \emptyset' = \mathfrak{I}'(\hbar) \cup \mathrm{U} = \mathrm{U}$. Thereby, $(\mathfrak{S}, \mathcal{K} \times \mathcal{K}) = \mathrm{U}_{\mathcal{K}\times\mathcal{K}}$. Similarly, $\emptyset_{\mathcal{K}}\mathrm{V}_*(\mathfrak{I}, \mathcal{K}) =_{\mathrm{M}} \mathrm{U}_{\mathcal{K}\times\mathcal{K}}$.

Proposition 7. Let $(\mathfrak{I}, \mathcal{K})$ be an SS over U. Then, $(\mathfrak{I}, \mathcal{K})\mathrm{V}_*\mathrm{U}_{\mathcal{K}} =_{\mathrm{M}} \mathrm{U}_{\mathcal{K}}\mathrm{V}_*(\mathfrak{I}, \mathcal{K}) =_{\mathrm{M}} (\mathfrak{I}, \mathcal{K}\mathrm{x}\mathcal{K})^{\mathrm{r}}$.

Proof: let $\mathrm{U}_{\mathcal{K}} = (\mathfrak{Q}, \mathcal{K})$. Then, for all $\hbar \in \mathcal{K}$, $\mathfrak{Q}(\hbar) = \mathrm{U}$. Let $(\mathfrak{I}, \mathcal{K})\mathrm{V}_*\mathrm{U}_{\mathcal{K}} = (\mathfrak{I}, \mathcal{K})\mathrm{V}_*(\mathfrak{Q}, \mathcal{K}) = (\mathfrak{S}, \mathcal{K} \times \mathcal{K})$, where for all $(\hbar, z) \in \mathcal{K} \times \mathcal{K}$, $\mathfrak{S}(\hbar, z) = \mathfrak{I}'(\hbar) \cup \mathfrak{Q}'(z) = \mathfrak{I}'(\hbar) \cup \mathrm{U}' = \mathfrak{I}'(\hbar) \cup \emptyset = \mathfrak{I}'(\hbar)$, implying that $(\mathfrak{S}, \mathcal{K} \times \mathcal{K}) = (\mathfrak{I}, \mathcal{K}\mathrm{x}\mathcal{K})^{\mathrm{r}}$. Similarly, $(\mathfrak{I}, \mathcal{K})\mathrm{V}_*(\mathfrak{Q}, \mathcal{K}) = (\mathfrak{I}, \mathcal{K}\mathrm{x}\mathcal{K})^{\mathrm{r}}$.

Proposition 8. Let $(\mathfrak{J},\mathcal{K})$ be an SS over U. Then, $(\mathfrak{J},\mathcal{K})^r \mathrel{\tilde{\subseteq}}_J (\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{J},\mathcal{K})$. That is, $\vee_*$-product is not idempotent in $S_E(U)$ under J-equality.

Proof: let $(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{J},\mathcal{K}) = (\mathfrak{S},\mathcal{K}\times\mathcal{K})$. Then, for all $(k,z)\in\mathcal{K}x\mathcal{K}$, $\mathfrak{S}(k,z)=\mathfrak{J}'(k)\cup\mathfrak{Q}'(z)$.

Since for all $k\in\mathcal{K}$, there exists $(k,z)\in\mathcal{K}x\mathcal{K}$ such that $\mathfrak{J}'(k)\subseteq\mathfrak{S}(k,z)=\mathfrak{J}'(k)\cup\mathfrak{J}'(z)$, $(\mathfrak{J},\mathcal{K})^r \mathrel{\tilde{\subseteq}}_J (\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{J},\mathcal{K})$ is obtained.

Proposition 9. Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. Then, $(\mathfrak{S},\mathcal{Z})^r \mathrel{\tilde{\subseteq}}_J (\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{J},\mathcal{K})^r \mathrel{\tilde{\subseteq}}_J (\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})$.

Proof: let $(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K}\times\mathcal{Z})$, where for all $(k,z)\in\mathcal{K}\times\mathcal{Z}$, $\mathfrak{Q}(k,z)=\mathfrak{J}'(k)\cup\mathfrak{S}'(z)$. Since for all $z\in\mathcal{Z}$, there exists $(k,z)\in\mathcal{K}\times\mathcal{Z}$ such that $\mathfrak{S}'(z)\subseteq\mathfrak{J}'(k)\cup\mathfrak{S}'(z)$, $(\mathfrak{S},\mathcal{Z})^r \mathrel{\tilde{\subseteq}}_J (\mathfrak{J},\mathcal{K})\wedge_*(\mathfrak{S},\mathcal{Z})$. Similarly, since for all $k\in\mathcal{K}$, there exists $(k,z)\in\mathcal{K}\times\mathcal{Z}$ such that $\mathfrak{J}'(k)\subseteq\mathfrak{J}'(k)\cup\mathfrak{S}'(z)$, $(\mathfrak{J},\mathcal{K})^r \mathrel{\tilde{\subseteq}}_J (\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})$.

Proposition 10. Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. Then, $[(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})]^r = (\mathfrak{J},\mathcal{K})^r \wedge_\theta (\mathfrak{S},\mathcal{Z})^r$.

Proof: let $(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K}\times\mathcal{Z})$, where for all $(k,z)\in\mathcal{K}\times\mathcal{Z}$, $\mathfrak{Q}(k,z)=\mathfrak{J}'(k)\cup\mathfrak{S}'(z)$. Thus, $\mathfrak{Q}'(k,z)=\mathfrak{J}(k)\cap\mathfrak{S}(z)=(\mathfrak{J}')'(k)\cap(\mathfrak{S}')'(z)$. Hence, $(\mathfrak{Q}',\mathcal{K}\times\mathcal{Z}) = (\mathfrak{J},\mathcal{K})^r \wedge_\theta (\mathfrak{S},\mathcal{Z})^r$. (For more about $\wedge_\theta$-product, please see [90].)

Proposition 11. Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. Then, $(\mathfrak{J},\mathcal{K})\wedge_\theta(\mathfrak{S},\mathcal{Z}) \mathrel{\tilde{\subseteq}}_F (\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})$.

Proof: let $(\mathfrak{J},\mathcal{K})\wedge_\theta(\mathfrak{S},\mathcal{Z}) = (\mathfrak{E},\mathcal{K}\times\mathcal{Z})$ and $(\mathfrak{J},\mathcal{K})\wedge_*(\mathfrak{S},\mathcal{Z}) = (\zeta,\mathcal{K}\times\mathcal{Z})$, where for all $(k,z)\in\mathcal{K}\times\mathcal{Z}$, $\mathfrak{E}(k,z)=\mathfrak{J}'(k)\cap\mathfrak{S}'(z)$ and for all $(k,z)\in\mathcal{K}\times\mathcal{Z}$, $\zeta(k,z)=\mathfrak{J}'(k)\cup\mathfrak{S}'(z)$. Since $\mathfrak{E}(k,z)=\mathfrak{J}'(k)\cap\mathfrak{S}'(z)\subseteq\mathfrak{J}'(k)\cup\mathfrak{S}'(z)=\zeta(k,z)$ for all $(k,z)\in\mathcal{K}\times\mathcal{Z}$, the proof is completed.

Proposition 12. Let $(\mathfrak{J},\mathcal{K})$, $(\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{Q},\mathcal{C})$ be SSs over U. If $(\mathfrak{J},\mathcal{K})^r \mathrel{\tilde{\subseteq}}_F (\mathfrak{S},\mathcal{Z})^r$, then $(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{Q},\mathcal{C}) \mathrel{\tilde{\subseteq}}_F (\mathfrak{S},\mathcal{Z})\vee_*(\mathfrak{Q},\mathcal{C})$.

Proof: let $(\mathfrak{J},\mathcal{K})^r \mathrel{\tilde{\subseteq}}_F (\mathfrak{S},\mathcal{Z})^r$. Then, $\mathcal{K}\subseteq\mathcal{Z}$ and for all $z\in\mathcal{Z}$, $\mathfrak{J}'(z)\subseteq\mathfrak{S}'(z)$. Thus, $\mathcal{K}\times\mathcal{C}\subseteq\mathcal{Z}\times\mathcal{C}$ and for all $(k,c)\in\mathcal{K}\times\mathcal{C}$, $\mathfrak{J}'(k)\cup\mathfrak{Q}'(c)\subseteq\mathfrak{S}'(k)\cup\mathfrak{Q}'(c)$. This completes the proof.

Proposition 13. Let $(\mathfrak{J},\mathcal{K}),(\mathfrak{S},\mathcal{Z}),(\mathfrak{Q},\mathcal{C})$ and $(\mathfrak{X},\mathcal{W})$ be SSs over U. If $(\mathfrak{J},\mathcal{K})^r \mathrel{\tilde{\subseteq}}_F (\mathfrak{S},\mathcal{Z})^r$ and $(\mathfrak{Q},\mathcal{C})^r \mathrel{\tilde{\subseteq}}_F (\mathfrak{X},\mathcal{W})^r$, then $(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{Q},\mathcal{C}) \mathrel{\tilde{\subseteq}}_F (\mathfrak{S},\mathcal{Z})\vee_*(\mathfrak{X},\mathcal{W})$ and $(\mathfrak{Q},\mathcal{C})\vee_*(\mathfrak{J},\mathcal{K}) \mathrel{\tilde{\subseteq}}_F (\mathfrak{X},\mathcal{W})\vee_*(\mathfrak{S},\mathcal{Z})$.

Proof: let $(\mathfrak{J},\mathcal{K})^r \mathrel{\tilde{\subseteq}}_F (\mathfrak{S},\mathcal{Z})^r$ and $(\mathfrak{Q},\mathcal{C})^r \mathrel{\tilde{\subseteq}}_F (\mathfrak{X},\mathcal{W})^r$. Then, $\mathcal{K}\subseteq\mathcal{Z}$, $\mathcal{C}\subseteq\mathcal{W}$, for all $k\in\mathcal{K}$, $\mathfrak{J}'(k)\subseteq\mathfrak{S}'(k)$ and for all $c\in\mathcal{C}$, $\mathfrak{Q}'(c)\subseteq\mathfrak{X}'(c)$. Thus, $\mathcal{K}\times\mathcal{C}\subseteq\mathcal{Z}\times\mathcal{W}$, for all $(k,c)\in\mathcal{K}\times\mathcal{C}$, $\mathfrak{J}'(k)\cup\mathfrak{Q}'(c)\subseteq\mathfrak{S}'(k)\cup\mathfrak{Q}'(c)$ and for all $(c,k)\in\mathcal{C}\times\mathcal{K}$, $\mathfrak{Q}'(c)\cup\mathfrak{J}'(k)\subseteq\mathfrak{X}'(c)\cup\mathfrak{S}'(k)$. This completes the proof.

Proposition 14. Let $(\mathfrak{J},\mathcal{K}),(\mathfrak{S},\mathcal{K}),(\mathfrak{Q},\mathcal{K})$ and $(\mathfrak{X},\mathcal{K})$ be SSs over U. If $(\mathfrak{J},\mathcal{K}) \mathrel{\tilde{\subseteq}}_F (\mathfrak{S},\mathcal{K})$ and $(\mathfrak{Q},\mathcal{K}) \mathrel{\tilde{\subseteq}}_F (\mathfrak{X},\mathcal{K})$, then $(\mathfrak{S},\mathcal{K})\vee_*(\mathfrak{X},\mathcal{K}) \mathrel{\tilde{\subseteq}}_F (\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{Q},\mathcal{K})$.

Proof: let $(\mathfrak{J},\mathcal{K}) \mathrel{\tilde{\subseteq}}_F (\mathfrak{S},\mathcal{K})$ and $(\mathfrak{Q},\mathcal{K}) \mathrel{\tilde{\subseteq}}_F (\mathfrak{X},\mathcal{K})$. Thus, for all $k\in\mathcal{K}$, $\mathfrak{J}(k)\subseteq\mathfrak{S}(k)$ and for all $\ell\in\mathcal{K}$, $\mathfrak{Q}(\ell)\subseteq\mathfrak{X}(\ell)$. Thereby, for all $(k,\ell)\in\mathcal{K}\times\mathcal{K}$, $\mathfrak{S}'(k)\cup\mathfrak{X}'(\ell)\subseteq\mathfrak{J}'(k)\cup\mathfrak{Q}'(\ell)$, completing the proof.

Proposition 15. Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. Then, $\emptyset_{\mathcal{K}\times\mathcal{Z}} \mathrel{\tilde{\subseteq}}_F (\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})$ and $\emptyset_{\mathcal{Z}\times\mathcal{K}} \mathrel{\tilde{\subseteq}}_F (\mathfrak{S},\mathcal{Z})\vee_*(\mathfrak{J},\mathcal{K})$.

Proof: let $\emptyset_{\mathcal{K}\times\mathcal{Z}} = (\mathcal{E},\mathcal{K}\times\mathcal{Z})$ and $(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z}) = (\mathcal{S},\mathcal{K}\times\mathcal{Z})$, where for all $(k,z)\in\mathcal{K}\times\mathcal{Z}$, $\mathcal{E}(k,z)=\emptyset$ and for all $(k,z)\in\mathcal{K}\times\mathcal{Z}$, $\mathcal{S}(k,z)=\mathfrak{J}'(k)\cup\mathfrak{S}'(z)$. Since $\mathcal{K}\times\mathcal{Z}\subseteq\mathcal{K}\times\mathcal{Z}$ and for all $(k,z)\in\mathcal{K}\times\mathcal{Z}$, $\mathcal{S}(k,z)=\emptyset\subseteq\mathfrak{J}'(k)\cup\mathfrak{S}'(z)=\mathcal{E}(k,z)$, $\emptyset_{\mathcal{K}\times\mathcal{Z}} \mathrel{\tilde{\subseteq}}_F (\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})$ is obtained. Similarly, $\emptyset_{\mathcal{Z}\times\mathcal{K}} \mathrel{\tilde{\subseteq}}_F (\mathfrak{S},\mathcal{Z})\vee_*(\mathfrak{J},\mathcal{K})$.

Proposition 16. Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. Then, $\emptyset_{\mathcal{K}} \mathrel{\tilde{\subseteq}}_J (\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})$ and $\emptyset_{\mathcal{K}} \mathrel{\tilde{\subseteq}}_J (\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})$ and $\emptyset_E \mathrel{\tilde{\subseteq}}_J (\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})$.

Proof: let $\emptyset_{\mathcal{K}} = (\mathcal{E}, \mathcal{K})$ and $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) = (\mathcal{S}, \mathcal{K} \times \mathcal{Z})$, where for all $\mathscr{k} \in \mathcal{K}$, $\mathcal{E}(\mathscr{k}) = \emptyset$ and for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{S}(\mathscr{k}, z) = \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{S}'(z)$. Since for all $\mathscr{k} \in \mathcal{K}$, there exists $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$ such that $\mathcal{E}(\mathscr{k}) = \emptyset \subseteq \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{S}'(z)\mathcal{S}(\mathscr{k}, z)$, $\emptyset_{\mathcal{K}} \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z})$ is obtained. Similarly, $\emptyset_{\mathcal{K}} \tilde{\subseteq}_{\mathrm{J}}(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z})$ and $\emptyset_{\mathrm{E}} \tilde{\subseteq}_{\mathrm{J}}(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z})$ are obtained.

Proposition 17. Let $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ be SSs over U. Then, $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_{\mathrm{F}} \mathrm{U}_{\mathcal{K}\times\mathcal{Z}}$ and $(\mathfrak{S}, \mathcal{Z})\mathrm{V}_*(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_{\mathrm{F}} \mathrm{U}_{\mathcal{Z}\times\mathcal{K}}$.

Proof: let $\mathrm{U}_{\mathcal{K}\times\mathcal{Z}} = (\mathfrak{Q}, \mathcal{K} \times \mathcal{Z})$ and $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) = (\mathfrak{A}, \mathcal{K} \times \mathcal{Z})$, where for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{Q}(\mathscr{k}, z) = \mathrm{U}$ and for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{A}(\mathscr{k}, z) = \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{S}'(z)$. Since $\mathcal{K} \times \mathcal{Z} \subseteq \mathcal{K} \times \mathcal{Z}$ and for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{A}(\mathscr{k}, z) = \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{S}'(z) \subseteq \mathrm{U} = \mathfrak{Q}(\mathscr{k}, z)$, $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_{\mathrm{F}} \mathrm{U}_{\mathcal{K}\times\mathcal{Z}}$ is obtained. Similarly, $(\mathfrak{S}, \mathcal{Z})\mathrm{V}_*(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_{\mathrm{F}} \mathrm{U}_{\mathcal{Z}\times\mathcal{K}}$.

Proposition 18. Let $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ be SSs over U. Then, $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_{\mathrm{J}} \mathrm{U}_{\mathcal{K}}$ and $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_{\mathrm{J}} \mathrm{U}_{\mathcal{Z}}$.

Proof: let $\mathrm{U}_{\mathcal{K}} = (\mathcal{W}, \mathcal{K})$ and $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) = (\mathcal{X}, \mathcal{K} \times \mathcal{Z})$, where for all $\mathscr{k} \in \mathcal{K}$, $\mathcal{W}(\mathscr{k}) = \mathrm{U}$ and for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{X}(\mathscr{k}, z) = \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{S}'(z)$. Since for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, there exists $\mathscr{k} \in \mathcal{K}$ such that $\mathcal{X}(\mathscr{k}, z) = \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{S}'(z) \subseteq \mathrm{U} = \mathcal{W}(\mathscr{k})$, $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_{\mathrm{J}} \mathrm{U}_{\mathcal{K}}$. Similarly, $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_{\mathrm{J}} \mathrm{U}_{\mathcal{Z}}$.

Proposition 19. Let $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ be SSs over U. Then, $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) =_{\mathrm{M}} \emptyset_{\mathcal{K}\times\mathcal{Z}}$ if and only if $(\mathfrak{J}, \mathcal{K}) =_{\mathrm{M}} \mathrm{U}_{\mathcal{K}}$ ve $(\mathfrak{S}, \mathcal{Z}) =_{\mathrm{M}} \mathrm{U}_{\mathcal{Z}}$.

Proof: let $\emptyset_{\mathcal{K}\times\mathcal{Z}} = (\mathcal{E}, \mathcal{K} \times \mathcal{Z})$ and $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) = (\mathcal{X}, \mathcal{K} \times \mathcal{Z})$, where for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{E}(\mathscr{k}, z) = \emptyset$, and for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{X}(\mathscr{k}, z) = \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{S}'(z)$. Let $(\mathcal{E}, \mathcal{K} \times \mathcal{Z}) = (\mathcal{X}, \mathcal{K} \times \mathcal{Z})$. Then, for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{J}'(\mathscr{k}) \cup \mathfrak{S}'(z) = \emptyset$. Thereby, for all $\mathscr{k} \in \mathcal{K}$, $\mathfrak{J}'(\mathscr{k}) = \emptyset$ and for all $z \in \mathcal{Z}$, $\mathfrak{S}'(z) = \emptyset$. Hence, for all $\mathscr{k} \in \mathcal{K}$, $\mathfrak{J}(\mathscr{k}) = \mathrm{U}$ and for all $z \in \mathcal{Z}$, $\mathfrak{S}(z) = \mathrm{U}$, implying that $(\mathfrak{J}, \mathcal{K}) =_{\mathrm{M}} \mathrm{U}_{\mathcal{K}}$ and $(\mathfrak{S}, \mathcal{Z}) =_{\mathrm{M}} \mathrm{U}_{\mathcal{Z}}$.

Conversely, let $(\mathfrak{J}, \mathcal{K}) =_{\mathrm{M}} \mathrm{U}_{\mathcal{K}}$ and $(\mathfrak{S}, \mathcal{Z}) =_{\mathrm{M}} \mathrm{U}_{\mathcal{Z}}$. Thus, for all $\mathscr{k} \in \mathcal{K}$, $\mathfrak{J}(\mathscr{k}) = \mathrm{U}$ and for all $z \in \mathcal{Z}$, $\mathfrak{S}(z) = \mathrm{U}$. Hence, for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{X}(\mathscr{k}, z) = \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{S}'(z) = \emptyset \cup \emptyset = \emptyset$, implying that $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) =_{\mathrm{M}} \emptyset_{\mathcal{K}\times\mathcal{Z}}$.

Proposition 20. Let $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ be SSs over U. Then, $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) =_{\mathrm{M}} \emptyset_{\emptyset}$ if and only if $(\mathfrak{J}, \mathcal{K}) =_{\mathrm{M}} \emptyset_{\emptyset}$ or $(\mathfrak{S}, \mathcal{Z}) =_{\mathrm{M}} \emptyset_{\emptyset}$.

Proof: let $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) =_{\mathrm{M}} \emptyset_{\emptyset}$. Thereby, $\mathcal{K} \times \mathcal{Z} = \emptyset$, and so $\mathcal{K} = \emptyset$ or $\mathcal{Z} = \emptyset$. Since $\emptyset_{\emptyset}$ is the only SS with the empty PS, $(\mathfrak{J}, \mathcal{K}) =_{\mathrm{M}} \emptyset_{\emptyset}$ or $(\mathfrak{S}, \mathcal{Z}) =_{\mathrm{M}} \emptyset_{\emptyset}$.

Conversely, let $(\mathfrak{J}, \mathcal{K}) = \emptyset_{\emptyset}$ or $(\mathfrak{S}, \mathcal{Z}) = \emptyset_{\emptyset}$. Then, $\mathcal{K} = \emptyset$ or $\mathcal{Z} = \emptyset$. Since $\mathcal{K} \times \mathcal{Z} = \emptyset$ and $\emptyset_{\emptyset}$ is the only SS with an empty PS, $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z}) =_{\mathrm{M}} \emptyset_{\emptyset}$.

4 | Distributions of Soft Star-Product Over Certain Types of Soft Set

In this section, we investigate the distributions of soft star-product over restricted, extended, soft binary piecewise intersection and union operations, AND-product and OR-product.

Theorem 1. Let $(\mathfrak{J}, \mathcal{K}), (\mathfrak{S}, \mathcal{Z})$ and $(\mathfrak{Q}, \mathcal{C})$ be SSs over U. Then, we have the following distributions of soft star-product over restricted intersection and union operations:

I. $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*[(\mathfrak{S}, \mathcal{Z}) \cup_{\mathrm{R}} (\mathfrak{Q}, \mathcal{C})] =_{\mathrm{M}} [(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z})] \cap_{\mathrm{R}} [(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{Q}, \mathcal{C})]$.

II. $(\mathfrak{J}, \mathcal{K})\mathrm{V}_*[(\mathfrak{S}, \mathcal{Z}) \cap_{\mathrm{R}} (\mathfrak{Q}, \mathcal{C})] =_{\mathrm{M}} [(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{S}, \mathcal{Z})] \cup_{\mathrm{R}} [(\mathfrak{J}, \mathcal{K})\mathrm{V}_*(\mathfrak{Q}, \mathcal{C})]$.

III. $[(\mathfrak{S}, \mathcal{Z}) \cap_{\mathrm{R}} (\mathfrak{Q}, \mathcal{C})]\mathrm{V}_*(\mathfrak{J}, \mathcal{K}) =_{\mathrm{M}} [(\mathfrak{S}, \mathcal{Z})\mathrm{V}_*(\mathfrak{J}, \mathcal{K})] \cup_{\mathrm{R}} [(\mathfrak{Q}, \mathcal{C})\mathrm{V}_*(\mathfrak{J}, \mathcal{K})]$.

IV. $[(\mathfrak{S}, \mathcal{Z}) \cup_{\mathrm{R}} (\mathfrak{Q}, \mathcal{C})]\mathrm{V}_*(\mathfrak{J}, \mathcal{K}) =_{\mathrm{M}} [(\mathfrak{S}, \mathcal{Z})\mathrm{V}_*(\mathfrak{J}, \mathcal{K})] \cap_{\mathrm{R}} [(\mathfrak{Q}, \mathcal{C})\mathrm{V}_*(\mathfrak{J}, \mathcal{K})]$.

Proof: the PS of the left-hand side (LHS) is $\mathcal{K}\mathrm{x}(\mathcal{Z} \cap \mathcal{C})$, and the PS of the right-hand side (RHS) is $(\mathcal{K}\mathrm{x}\mathcal{Z}) \cap (\mathcal{K}\mathrm{x}\mathcal{C})$. Since $\mathcal{K}\mathrm{x}(\mathcal{Z} \cap \mathcal{C}) = (\mathcal{K}\mathrm{x}\mathcal{Z}) \cap (\mathcal{K}\mathrm{x}\mathcal{C})$, the first condition of the M-equality is satisfied. Let $(\mathfrak{S},\mathcal{Z}) \cup_{\mathrm{R}} (\mathfrak{Q},\mathcal{C}) = (\mathfrak{E},\mathcal{Z} \cap \mathcal{C})$, where for all $\mathfrak{z} \in \mathcal{Z} \cap \mathcal{C}$, $\mathfrak{E}(\mathfrak{z}) = \mathfrak{S}(\mathfrak{z}) \cup \mathfrak{Q}(\mathfrak{z})$. Let $(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{E},\mathcal{Z} \cap \mathcal{C}) = (\wp,\mathcal{K} \times (\mathcal{Z} \cap \mathcal{C}))$, where for all $(\mathscr{k},\mathfrak{z}) \in \mathcal{K} \times (\mathcal{Z} \cap \mathcal{C})$, $\wp(\mathscr{k},\mathfrak{z}) = \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{E}'(\mathfrak{z})$. Then,

$$\wp(\mathscr{k},\mathfrak{z}) = \mathfrak{J}'(\mathscr{k}) \cup [\mathfrak{S}(\mathfrak{z}) \cup \mathfrak{Q}(\mathfrak{z})]' = \mathfrak{J}'(\mathscr{k}) \cup [\mathfrak{S}'(\mathfrak{z}) \cap \mathfrak{Q}'(\mathfrak{z})].$$

Let $(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{S},\mathcal{Z}) = (\mathfrak{M},\mathcal{K} \times \mathcal{Z})$ and $(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{Q},\mathcal{C}) = (\mathfrak{P},\mathcal{K} \times \mathcal{C})$, where for all $(\mathscr{k},z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{M}(\mathscr{k},z) = \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{S}'(z)$ and for all $(\mathscr{k},c) \in \mathcal{K} \times \mathcal{C}$, $\mathfrak{P}(\mathscr{k},c) = \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{Q}'(c)$. Thus, $(\mathfrak{M},\mathcal{K} \times \mathcal{Z}) \cap_{\mathrm{R}} (\mathfrak{P},\mathcal{K} \times \mathcal{C}) = (\mathfrak{R},(\mathcal{K} \times \mathcal{Z}) \cap (\mathcal{K} \times \mathcal{C}))$, where for all $(\mathscr{k},\mathfrak{z}) \in (\mathcal{K} \times \mathcal{Z}) \cap (\mathcal{K} \times \mathcal{C}) = \mathcal{K} \times (\mathcal{Z} \cap \mathcal{C})$,

$$\mathfrak{R}(\mathscr{k},\mathfrak{z}) = \mathfrak{M}(\mathscr{k},\mathfrak{z}) \cap \mathfrak{P}(\mathscr{k},\mathfrak{z}) = [\mathfrak{J}'(\mathscr{k}) \cup \mathfrak{S}'(\mathfrak{z})] \cap [\mathfrak{J}'(\mathscr{k}) \cup \mathfrak{Q}'(\mathfrak{z})].$$

Therefore, $(\mathfrak{J},\mathcal{K})\mathrm{V}_*[(\mathfrak{S},\mathcal{Z}) \cup_{\mathrm{R}} (\mathfrak{Q},\mathcal{C})] =_{\mathrm{M}} [(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{S},\mathcal{Z})] \cap_{\mathrm{R}} [(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{Q},\mathcal{C})]$.

Here, if $\mathcal{Z} \cap \mathcal{C} = \emptyset$, then $\mathcal{K}\mathrm{x}(\mathcal{Z} \cap \mathcal{C}) = (\mathcal{K}\mathrm{x}\mathcal{Z}) \cap (\mathcal{K}\mathrm{x}\mathcal{C}) = \emptyset$. Since the only SS with an empty PS is $\emptyset_\emptyset$, then both sides are $\emptyset_\emptyset$. Since $(\mathcal{K}\mathrm{x}\mathcal{Z}) \cap (\mathcal{K}\mathrm{x}\mathcal{C}) = \mathcal{K}\mathrm{x}(\mathcal{Z} \cap \mathcal{C})$, if $(\mathcal{K}\mathrm{x}\mathcal{Z}) \cap (\mathcal{K}\mathrm{x}\mathcal{C})=\emptyset$, then $\mathcal{K}=\emptyset$ or $\mathcal{Z} \cap \mathcal{C} = \emptyset$. By assumption, $\mathcal{K} \neq \emptyset$. Thus, $(\mathcal{K}\mathrm{x}\mathcal{Z}) \cap (\mathcal{K}\mathrm{x}\mathcal{C})=\emptyset$ implies that $\mathcal{Z} \cap \mathcal{C} = \emptyset$. Therefore, under this condition, both sides are again $\emptyset_\emptyset$.

The PS of the LHS is $(\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K}$, and the PS of the RHS is $(\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K})$, and since $(\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K} = (\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K})$, the first condition of M-equality is satisfied. Let $(\mathfrak{S},\mathcal{Z}) \cap_{\mathrm{R}} (\mathfrak{Q},\mathcal{C}) = (\mathfrak{E},\mathcal{Z} \cap \mathcal{C})$, where for all $\mathfrak{z} \in \mathcal{Z} \cap \mathcal{C}$, $\mathfrak{E}(\mathfrak{z}) = \mathfrak{S}(\mathfrak{z}) \cap \mathfrak{Q}(\mathfrak{z})$. Let $(\mathfrak{E},\mathcal{Z} \cap \mathcal{C})\mathrm{V}_*(\mathfrak{J},\mathcal{K}) = (\wp,(\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K})$, where for all $(\mathfrak{z},\mathscr{k}) \in (\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K}$, $\wp(\mathfrak{z},\mathscr{k}) = \mathfrak{E}'(\mathfrak{z}) \cup \mathfrak{J}'(\mathscr{k})$. Thus,

$$\wp(\mathfrak{z},\mathscr{k}) = [\mathfrak{S}(\mathfrak{z}) \cap \mathfrak{Q}(\mathfrak{z})]' \cup \mathfrak{J}'(\mathscr{k}) = [\mathfrak{S}'(\mathfrak{z}) \cup \mathfrak{Q}'(\mathfrak{z})] \cup \mathfrak{J}'(\mathscr{k}).$$

Let $(\mathfrak{S},\mathcal{Z})\mathrm{V}_*(\mathfrak{J},\mathcal{K}) = (\mathfrak{M},\mathcal{Z} \times \mathcal{K})$ and $(\mathfrak{Q},\mathcal{C})\mathrm{V}_*(\mathfrak{J},\mathcal{K}) = (\mathfrak{P},\mathcal{C} \times \mathcal{K})$, where for all $(z,\mathscr{k}) \in \mathcal{Z} \times \mathcal{K}$, $\mathfrak{M}(z,\mathscr{k}) = \mathfrak{S}'(\mathfrak{z}) \cup \mathfrak{J}'(\mathscr{k})$ and for all $(c,\mathscr{k}) \in \mathcal{C} \times \mathcal{K}$, $\mathfrak{P}(c,\mathscr{k}) = \mathfrak{Q}'(c) \cup \mathfrak{J}'(\mathscr{k})$. Assume that $(\mathfrak{M},\mathcal{Z} \times \mathcal{K}) \cup_{\mathrm{R}} (\mathfrak{P},\mathcal{C} \times \mathcal{K}) = (\mathfrak{R},(\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}))$, where for all $(\mathfrak{z},\mathscr{k}) \in (\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}) = (\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K}$,

$$\mathfrak{R}(\mathfrak{z},\mathscr{k}) = \mathfrak{M}(\mathfrak{z},\mathscr{k}) \cup \mathfrak{P}(\mathfrak{z},\mathscr{k}) = [\mathfrak{S}'(\mathfrak{z}) \cup \mathfrak{J}'(\mathscr{k})] \cup [\mathfrak{Q}'(\mathfrak{z}) \cup \mathfrak{J}'(\mathscr{k})].$$

Thereby, $[(\mathfrak{S},\mathcal{Z}) \cap_{\mathrm{R}} (\mathfrak{Q},\mathcal{C})]\mathrm{V}_*(\mathfrak{J},\mathcal{K}) =_{\mathrm{M}} (\mathfrak{S},\mathcal{Z})\mathrm{V}_*(\mathfrak{J},\mathcal{K}) \cup_{\mathrm{R}} (\mathfrak{Q},\mathcal{C})\mathrm{V}_*(\mathfrak{J},\mathcal{K})$.

Here, if $\mathcal{Z} \cap \mathcal{C} = \emptyset$, then $(\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K} = (\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K})=\emptyset$. Since the only SS with the empty parameter set is $\emptyset_\emptyset$, then both sides of the equality are $\emptyset_\emptyset$. Similarly since $(\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}) = (\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K}$, if $(\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}) = \emptyset$, then $\mathcal{Z} \cap \mathcal{C} = \emptyset$ or $\mathcal{K} = \emptyset$. By assumption $\mathcal{K} \neq \emptyset$. Hence, $(\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}) = \emptyset$ implies that $\mathcal{Z} \cap \mathcal{C} = \emptyset$. Thus, under this condition, both sides of the equality are again $\emptyset_\emptyset$.

Note 1: the restricted SS operation can not distribute over soft star-product as the intersection does not distribute over cartesian product, and it is compulsory for two SSs to be M-equal that their PS should be the same.

Theorem 2. Let $(\mathfrak{J},\mathcal{K}), (\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{Q},\mathcal{C})$ be SSs over **U**. Then, we have the following distributions of soft star-product over extended intersection and union operations:

I. $(\mathfrak{J},\mathcal{K})\mathrm{V}_*[(\mathfrak{S},\mathcal{Z}) \cap_\varepsilon (\mathfrak{Q},\mathcal{C})] =_{\mathrm{M}} [(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{S},\mathcal{Z})] \cup_\varepsilon [(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{Q},\mathcal{C})]$.

II. $(\mathfrak{J},\mathcal{K})\mathrm{V}_*[(\mathfrak{S},\mathcal{Z}) \cup_\varepsilon (\mathfrak{Q},\mathcal{C})] =_{\mathrm{M}} [(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{S},\mathcal{Z})] \cap_\varepsilon [(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{Q},\mathcal{C})]$.

III. $[(\mathfrak{S},\mathcal{Z}) \cup_\varepsilon (\mathfrak{Q},\mathcal{C})]\mathrm{V}_*(\mathfrak{J},\mathcal{K}) =_{\mathrm{M}} [(\mathfrak{S},\mathcal{Z})\mathrm{V}_*(\mathfrak{J},\mathcal{K})] \cap_\varepsilon [(\mathfrak{Q},\mathcal{C})\mathrm{V}_*(\mathfrak{J},\mathcal{K})]$.

IV. $[(\mathfrak{S},\mathcal{Z}) \cap_\varepsilon (\mathfrak{Q},\mathcal{C})]\mathrm{V}_*(\mathfrak{J},\mathcal{K}) =_{\mathrm{M}} [(\mathfrak{S},\mathcal{Z})\mathrm{V}_*(\mathfrak{J},\mathcal{K})] \cup_\varepsilon [(\mathfrak{Q},\mathcal{C})\mathrm{V}_*(\mathfrak{J},\mathcal{K})]$.

Proof: the PS of LHS is $\mathcal{K}\mathrm{x}(\mathcal{Z} \cup \mathcal{C})$, and the PS of the RHS is $(\mathcal{K}\mathrm{x}\mathcal{Z}) \cup (\mathcal{K}\mathrm{x}\mathcal{C})$. Since $\mathcal{K}\mathrm{x}(\mathcal{Z} \cup \mathcal{C}) = (\mathcal{K}\mathrm{x}\mathcal{Z}) \cup (\mathcal{K}\mathrm{x}\mathcal{C})$, the first condition of the M-equality is satisfied. As $\mathcal{K} \neq \emptyset$, $\mathcal{Z} \neq \emptyset$ and $\mathcal{J} \neq \emptyset$, $\mathcal{K}\mathrm{x}(\mathcal{Z} \cup \mathcal{C}) \neq \emptyset$ and $(\mathcal{K}\mathrm{x}\mathcal{Z}) \cup (\mathcal{K}\mathrm{x}\mathcal{C}) \neq \emptyset$. No side can, therefore, be equivalent to an empty SS. Let $(\mathfrak{S},\mathcal{Z}) \cap_\varepsilon (\mathfrak{Q},\mathcal{C}) = (\mathfrak{E},\mathcal{Z} \cup \mathcal{C})$, where for all $\mathfrak{z} \in \mathcal{Z} \cup \mathcal{C}$,

$$\mathfrak{E}(\mathfrak{z}) = \begin{cases} \mathfrak{S}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{Z} - \mathcal{C}, \\ \mathfrak{Q}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{C} - \mathcal{Z}, \\ \mathfrak{S}(\mathfrak{z}) \cap \mathfrak{Q}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{Z} \cap \mathcal{C}. \end{cases}$$

Let $(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{E},\mathcal{Z} \cup \mathcal{C}) = (\mathbb{Q},\mathcal{K} \times (\mathcal{Z} \cup \mathcal{C}))$, where for all $(\mathscr{k},\mathfrak{z}) \in \mathcal{K} \times (\mathcal{Z} \cup \mathcal{C})$, $\mathbb{Q}(\mathscr{k},\mathfrak{z}) = \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{E}'(\mathfrak{z})$. Thus, for all $(\mathscr{k},\mathfrak{z}) \in \mathcal{K} \times (\mathcal{Z} \cup \mathcal{C})$,

$$\mathbb{Q}(\mathscr{k},\mathfrak{z}) = \begin{cases} \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{S}'(\mathfrak{z}), & (\mathscr{k},\mathfrak{z}) \in \mathcal{K} \times (\mathcal{Z} - \mathcal{C}) \\ \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{Q}'(\mathfrak{z}), & (\mathscr{k},\mathfrak{z}) \in \mathcal{K} \times (\mathcal{C} - \mathcal{Z}) \\ \mathfrak{J}'(\mathscr{k}) \cup [\mathfrak{S}'(\mathfrak{z}) \cup \mathfrak{Q}'(\mathfrak{z})], & (\mathscr{k},\mathfrak{z}) \in \mathcal{K} \times (\mathcal{Z} \cap \mathcal{C}) \end{cases}$$

Assume that $(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{S},\mathcal{Z}) = ((\mathfrak{M},\mathcal{K} \times \mathcal{Z}))$ and $(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{Q},\mathcal{C}) = (\mathfrak{P},\mathcal{K} \times \mathcal{C})$, where for all $(\mathscr{k},z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{M}(\mathscr{k},z) = \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{S}'(z)$ and for all $(\mathscr{k},c) \in \mathcal{K} \times \mathcal{C}$, $\mathfrak{P}(\mathscr{k},c) = \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{Q}'(c)$. Let $(\mathfrak{M},\mathcal{K} \times \mathcal{Z}) \cup_\varepsilon (\mathfrak{P},\mathcal{K} \times \mathcal{C}) = (\mathfrak{R},(\mathcal{K} \times \mathcal{Z}) \cup (\mathcal{K} \times \mathcal{C}))$, where for all $(\mathscr{k},\mathfrak{z}) \in (\mathcal{K} \times \mathcal{Z}) \cup (\mathcal{K} \times \mathcal{C}) = \mathcal{K} \times (\mathcal{Z} \cup \mathcal{C})$,

$$\mathfrak{R}(\mathscr{k},\mathfrak{z}) = \begin{cases} \mathfrak{M}(\mathscr{k},\mathfrak{z}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{K} \times \mathcal{Z}) - (\mathcal{K} \times \mathcal{C}) = \mathcal{K} \times (\mathcal{Z} - \mathcal{C}) \\ \mathfrak{P}(\mathscr{k},\mathfrak{z}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{K} \times \mathcal{C}) - (\mathcal{K} \times \mathcal{Z}) = \mathcal{K} \times (\mathcal{C} - \mathcal{Z}) \\ \mathfrak{M}(\mathscr{k},\mathfrak{z}) \cup \mathfrak{P}(\mathscr{k},\mathfrak{z}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{K} \times \mathcal{Z}) \cap (\mathcal{K} \times \mathcal{C}) = \mathcal{K} \times (\mathcal{Z} \cap \mathcal{C}) \end{cases}$$

Therefore,

$$\mathfrak{R}(\mathscr{k},\mathfrak{z}) = \begin{cases} \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{S}'(\mathfrak{z}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{K} \times \mathcal{Z}) - (\mathcal{K} \times \mathcal{C}) = \mathcal{K} \times (\mathcal{Z} - \mathcal{C}), \\ \mathfrak{J}'(\mathscr{k}) \cup \mathfrak{Q}'(\mathfrak{z}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{K} \times \mathcal{C}) - (\mathcal{K} \times \mathcal{Z}) = \mathcal{K} \times (\mathcal{C} - \mathcal{Z}), \\ [\mathfrak{J}'(\mathscr{k}) \cup \mathfrak{S}'(\mathfrak{z})] \cup [\mathfrak{J}'(\mathscr{k}) \cup \mathfrak{Q}'(\mathfrak{z})], & (\mathscr{k},\mathfrak{z}) \in (\mathcal{K} \times \mathcal{Z}) \cap (\mathcal{K} \times \mathcal{C}) = \mathcal{K} \times (\mathcal{Z} \cap \mathcal{C}). \end{cases}$$

Hence, $(\mathfrak{J},\mathcal{K})\mathrm{V}_*[(\mathfrak{S},\mathcal{Z}) \cap_\varepsilon (\mathfrak{Q},\mathcal{C})] =_\mathrm{M} [(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{S},\mathcal{Z})] \cup_\varepsilon [(\mathfrak{J},\mathcal{K})\mathrm{V}_*(\mathfrak{Q},\mathcal{C})]$.

The PS of the LHS is $(\mathcal{Z} \cup \mathcal{C}) \times \mathcal{K}$, and the PS of the RHS is $(\mathcal{Z} \times \mathcal{K}) \cup (\mathcal{C} \times \mathcal{K})$, and since $(\mathcal{Z} \cup \mathcal{C}) \times \mathcal{K} = (\mathcal{Z} \times \mathcal{K}) \cup (\mathcal{C} \times \mathcal{K})$, the first condition of M-equality is satisfied. By assumption, $\mathcal{K} \neq \emptyset$, $\mathcal{Z} \neq \emptyset$ and $\mathcal{C} \neq \emptyset$. Thus, $(\mathcal{Z} \cup \mathcal{C}) \times \mathcal{K} \neq \emptyset$ and $(\mathcal{Z} \times \mathcal{K}) \cup (\mathcal{C} \times \mathcal{K}) \neq \emptyset$. No side can, therefore, be equivalent to an empty SS. Let $(\mathfrak{S},\mathcal{Z}) \cup_\varepsilon (\mathfrak{Q},\mathcal{C}) = (\mathfrak{E},\mathcal{Z} \cup \mathcal{C})$, where for all $\mathfrak{z} \in \mathcal{Z} \cup \mathcal{C}$,

$$\mathfrak{E}(\mathfrak{z}) = \begin{cases} \mathfrak{S}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{Z} - \mathcal{C}, \\ \mathfrak{Q}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{C} - \mathcal{Z}, \\ \mathfrak{S}(\mathfrak{z}) \cup \mathfrak{Q}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{Z} \cap \mathcal{C}. \end{cases}$$

Let $(\mathfrak{E},\mathcal{Z} \cup \mathcal{C})\mathrm{V}_*(\mathfrak{J},\mathcal{K}) = (\wp,(\mathcal{Z} \cup \mathcal{C}) \times \mathcal{K})$, where for all $(\mathfrak{z},\mathscr{k}) \in (\mathcal{Z} \cup \mathcal{C}) \times \mathcal{K}$, $\wp(\mathfrak{z},\mathscr{k}) = \mathfrak{E}'(\mathfrak{z}) \cup \mathfrak{J}'(\mathscr{k})$,

$$\wp(\mathscr{k},\mathfrak{z}) = \begin{cases} \mathfrak{S}'(\mathfrak{z}) \cup \mathfrak{J}'(\mathscr{k}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{Z} - \mathcal{C}) \times \mathcal{K} \\ \mathfrak{Q}'(\mathfrak{z}) \cup \mathfrak{J}'(\mathscr{k}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{C} - \mathcal{Z}) \times \mathcal{K} \\ [\mathfrak{S}'(\mathfrak{z}) \cap \mathfrak{Q}'(\mathfrak{z})] \cup \mathfrak{J}'(\mathscr{k}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K} \end{cases}$$

Suppose that $(\mathfrak{S},\mathcal{Z})\mathrm{V}_*(\mathfrak{J},\mathcal{K}) = (\mathfrak{M},\mathcal{Z} \times \mathcal{K})$ and $(\mathfrak{Q},\mathcal{C})\mathrm{V}_*(\mathfrak{J},\mathcal{K}) = (\mathfrak{P},\mathcal{C} \times \mathcal{K})$, where for all $(z,\mathscr{k}) \in \mathcal{Z} \times \mathcal{K}$, $\mathfrak{M}(z,\mathscr{k}) = \mathfrak{S}'(\mathfrak{z}) \cup \mathfrak{J}'(\mathscr{k})$ and for all $(c,\mathscr{k}) \in \mathcal{C} \times \mathcal{K}$, $\mathfrak{P}(c,\mathscr{k}) = \mathfrak{Q}'(c) \cup \mathfrak{J}'(\mathscr{k})$. Assume that $(\mathfrak{M},\mathcal{Z} \times \mathcal{K}) \cap_\varepsilon (\mathfrak{P},\mathcal{C} \times \mathcal{K}) = (\mathfrak{R},(\mathcal{Z} \times \mathcal{K}) \cup (\mathcal{C} \times \mathcal{K}))$, where for all $(\mathfrak{z},\mathscr{k}) \in (\mathcal{Z} \times \mathcal{K}) \cup (\mathcal{C} \times \mathcal{K}) = (\mathcal{Z} \cup \mathcal{C}) \times \mathcal{K}$,

$$\mathfrak{R}(\mathscr{k},\mathfrak{z}) = \begin{cases} \mathfrak{M}(\mathfrak{z},\mathscr{k}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{Z} \times \mathcal{K}) - (\mathcal{C} \times \mathcal{K}) = (\mathcal{Z} - \mathcal{C}) \times \mathcal{K} \\ \mathfrak{P}(\mathfrak{z},\mathscr{k}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{C} \times \mathcal{K}) - (\mathcal{Z} \times \mathcal{K}) = (\mathcal{C} - \mathcal{Z}) \times \mathcal{K} \\ \mathfrak{M}(\mathfrak{z},\mathscr{k}) \cap \mathfrak{P}(\mathfrak{z},\mathscr{k}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}) = (\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K} \end{cases}$$

Thus,

$$\mathfrak{R}(\mathscr{k},\mathfrak{z}) = \begin{cases} \mathfrak{S}'(\mathfrak{z}) \cup \mathfrak{J}'(\mathscr{k}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{Z} \times \mathcal{K}) - (\mathcal{C} \times \mathcal{K}) = (\mathcal{Z} - \mathcal{C}) \times \mathcal{K}, \\ \mathfrak{Q}'(\mathfrak{z}) \cup \mathfrak{J}'(\mathscr{k}), & (\mathscr{k},\mathfrak{z}) \in (\mathcal{C} \times \mathcal{K}) - (\mathcal{Z} \times \mathcal{K}) = (\mathcal{C} - \mathcal{Z}) \times \mathcal{K}, \\ [\mathfrak{S}'(\mathfrak{z}) \cup \mathfrak{J}'(\mathscr{k})] \cap [\mathfrak{Q}'(\mathfrak{z}) \cup \mathfrak{J}'(\mathscr{k})], & (\mathscr{k},\mathfrak{z}) \in (\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}) = (\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K}. \end{cases}$$

Hence, $[(\mathfrak{S},\mathcal{Z}) \cup_\varepsilon (\mathfrak{Q},\mathcal{C})]\vee_*(\mathfrak{J},\mathcal{K}) =_M [(\mathfrak{S},\mathcal{Z})\vee_*(\mathfrak{J},\mathcal{K})] \cap_\varepsilon [(\mathfrak{Q},\mathcal{C})\vee_*(\mathfrak{J},\mathcal{K})]$.

Note: the extended SS operation can not distribute over soft star-product as the union operation does not distribute over cartesian product, and it is compulsory for two SSs to be M-equal that their PS should be the same.

Theorem 3. Let $(\mathfrak{J},\mathcal{K}), (\mathfrak{S},\mathcal{Z})$ and $(\mathcal{Q},\mathcal{C})$ be SSs over **U.** Then, we have the following distributions of soft star-product over soft binary piecewise intersection and union operations:

I. $(\mathfrak{J},\mathcal{K})\vee_*[(\mathfrak{S},\mathcal{Z}) \tilde{\cap} (\mathcal{Q},\mathcal{C})] =_M [(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})] \tilde{\cup} [(\mathfrak{J},\mathcal{K})\vee_*(\mathcal{Q},\mathcal{C})]$.

II. $(\mathfrak{J},\mathcal{K})\vee_*[(\mathfrak{S},\mathcal{Z}) \tilde{\cup} (\mathcal{Q},\mathcal{C})] =_M [(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})] \tilde{\cap} [(\mathfrak{J},\mathcal{K})\vee_*(\mathcal{Q},\mathcal{C})]$.

III. $[(\mathfrak{S},\mathcal{Z}) \tilde{\cup} (\mathfrak{Q},\mathcal{C})]\vee_*(\mathfrak{J},\mathcal{K}) =_M [(\mathfrak{S},\mathcal{Z})\vee_*(\mathfrak{J},\mathcal{K})] \tilde{\cap} [(\mathfrak{Q},\mathcal{C})\vee_*(\mathfrak{J},\mathcal{K})]$.

IV. $[(\mathfrak{S},\mathcal{Z}) \tilde{\cap} (\mathfrak{Q},\mathcal{C})]\vee_*(\mathfrak{J},\mathcal{K}) =_M [(\mathfrak{S},\mathcal{Z})\vee_*(\mathfrak{J},\mathcal{K})] \tilde{\cup} [(\mathfrak{Q},\mathcal{C})\vee_*(\mathfrak{J},\mathcal{K})]$.

Proof: since the PS of the SSs of both sides are $\mathcal{K}\mathrm{x}\mathcal{Z}$, the first condition of the M-equality is satisfied. Moreover, since $\mathcal{K} \neq \emptyset$ and $\mathcal{Z} \neq \emptyset$ by assumption, $\mathcal{K}\mathrm{x}\mathcal{Z} \neq \emptyset$. No side can, therefore, be equivalent to an empty SS. Let $(\mathfrak{S},\mathcal{Z}) \tilde{\cap} (\mathfrak{Q},\mathcal{C}) = (\mathfrak{E},\mathcal{Z})$, where for all $\mathfrak{z} \in \mathcal{Z}$,

$$\mathfrak{E}(\mathfrak{z}) = \begin{cases} \mathfrak{S}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{Z} - \mathcal{C}, \\ \mathfrak{S}(\mathfrak{z}) \cap \mathfrak{Q}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{Z} \cap \mathcal{C}. \end{cases}$$

Let $(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{E},\mathcal{Z}) = (\mathbb{Q},\mathcal{K}\times\mathcal{Z})$, where for all $(\mathcal{k},\mathfrak{z}) \in \mathcal{K}\times\mathcal{Z}$, $\mathbb{Q}(\mathcal{k},\mathfrak{z}) = \mathfrak{J}'(\mathcal{k}) \cup \mathfrak{E}'(\mathfrak{z})$. Thus,

$$\mathbb{Q}(\mathcal{k},\mathfrak{z}) = \begin{cases} \mathfrak{J}'(\mathcal{k}) \cup \mathfrak{S}'(\mathfrak{z}), & (\mathcal{k},\mathfrak{z}) \in \mathcal{K}\times(\mathcal{Z} - \mathcal{C}), \\ \mathfrak{J}'(\mathcal{k}) \cup [\mathfrak{S}'(\mathfrak{z}) \cup \mathfrak{Q}'(\mathfrak{z})], & (\mathcal{k},\mathfrak{z}) \in \mathcal{Z}\cap\mathcal{C}. \end{cases}$$

Let $(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z}) = (\mathfrak{M},\mathcal{K}\times\mathcal{Z})$ and $(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{Q},\mathcal{C}) = (\mathfrak{P},\mathcal{K}\times\mathcal{C})$, where for all $(\mathcal{k},z) \in \mathcal{K}\times\mathcal{Z}$, $\mathfrak{M}(\mathcal{k},z) = \mathfrak{J}'(\mathcal{k}) \cup \mathfrak{S}'(\mathfrak{z})$ and for all $(\mathcal{k},c) \in \mathcal{K}\times\mathcal{C}$, $\mathfrak{P}(\mathcal{k},c) = \mathfrak{J}'(\mathcal{k}) \cup \mathfrak{Q}'(c)$. Suppose that $(\mathfrak{M},\mathcal{K}\times\mathcal{Z}) \tilde{\cup} (\mathfrak{P},\mathcal{K}\times\mathcal{C}) = (\mathfrak{R},(\mathcal{K}\times\mathcal{Z}))$, where for all $(\mathcal{k},\mathfrak{z}) \in \mathcal{K}\times\mathcal{Z}$,

$$\mathfrak{R}(\mathcal{k},\mathfrak{z}) = \begin{cases} \mathfrak{M}(\mathcal{k},\mathfrak{z}), & (\mathcal{k},\mathfrak{z}) \in (\mathcal{K}\times\mathcal{Z}) - (\mathcal{K}\times\mathcal{C}) = \mathcal{K}\times(\mathcal{Z}-\mathcal{C}), \\ \mathfrak{M}(\mathcal{k},\mathfrak{z}) \cup \mathfrak{P}(\mathcal{k},\mathfrak{z}), & (\mathcal{k},\mathfrak{z}) \in (\mathcal{K}\times\mathcal{Z}) \cap (\mathcal{K}\times\mathcal{C}) = \mathcal{K}\times(\mathcal{Z}\cap\mathcal{C}). \end{cases}$$

Then

$$\mathfrak{R}(\mathcal{k},\mathfrak{z}) = \begin{cases} \mathfrak{J}'(\mathcal{k}) \cup \mathfrak{S}'(\mathfrak{z}), & (\mathcal{k},\mathfrak{z}) \in (\mathcal{K}\times\mathcal{Z}) - (\mathcal{K}\times\mathcal{C}) = \mathcal{K}\times(\mathcal{Z}-\mathcal{C}), \\ [\mathfrak{J}'(\mathcal{k}) \cup \mathfrak{S}'(\mathfrak{z})] \cup [\mathfrak{J}'(\mathcal{k}) \cup \mathfrak{Q}'(\mathfrak{z})], & (\mathcal{k},\mathfrak{z}) \in (\mathcal{K}\times\mathcal{Z}) \cap (\mathcal{K}\times\mathcal{C}) = \mathcal{K}\times(\mathcal{Z}\cap\mathcal{C}). \end{cases}$$

Hence, $(\mathfrak{J},\mathcal{K})\vee_*[(\mathfrak{S},\mathcal{Z}) \tilde{\cap} (\mathfrak{Q},\mathcal{C})] =_M [(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})] \tilde{\cup} [(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{Q},\mathcal{C})]$.

Since $\mathcal{K} \neq \mathcal{K}\mathrm{x}\mathcal{K}$, the soft binary piecewise operations do not distribute over soft star-product operations.

Since the PS of the SSs of both sides are $\mathcal{Z}\times\mathcal{K}$, the first condition of the M-equality is satisfied. Moreover, since $\mathcal{Z} \neq \emptyset$ and $\mathcal{K} \neq \emptyset$ by assumption, $\mathcal{Z}\times\mathcal{K} \neq \emptyset$. No side can, therefore, be equivalent to an empty SS. Let $(\mathfrak{S},\mathcal{Z}) \tilde{\cup} (\mathfrak{Q},\mathcal{C}) = (\mathfrak{E},\mathcal{Z})$, where for all $\mathfrak{z} \in \mathcal{Z}$,

$$\mathfrak{E}(\mathfrak{z}) = \begin{cases} \mathfrak{S}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{Z} - \mathcal{C}, \\ \mathfrak{S}(\mathfrak{z}) \cup \mathfrak{Q}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{Z} \cap \mathcal{C}. \end{cases}$$

Assume that $(\mathfrak{E},\mathcal{Z})\vee_*(\mathfrak{J},\mathcal{K}) = (\wp,\mathcal{Z}\times\mathcal{K})$, where for all $(\mathfrak{z},\mathcal{k}) \in \mathcal{Z}\times\mathcal{K}$, $\wp(\mathfrak{z},\mathcal{k}) = \mathfrak{E}'(\mathfrak{z}) \cup \mathfrak{J}'(\mathcal{k})$. Hence,

$$\wp(\mathcal{k},\mathfrak{z}) = \begin{cases} \mathfrak{S}'(\mathfrak{z}) \cup \mathfrak{J}'(\mathcal{k}), & (\mathcal{k},\mathfrak{z}) \in (\mathcal{Z}-\mathcal{C})\times\mathcal{K}, \\ [\mathfrak{S}'(\mathfrak{z}) \cap \mathfrak{Q}'(\mathfrak{z})] \cup \mathfrak{J}'(\mathcal{k}), & (\mathcal{k},\mathfrak{z}) \in (\mathcal{Z}\cap\mathcal{C})\times\mathcal{K}. \end{cases}$$

Let $(\mathfrak{S},\mathcal{Z})\vee_*(\mathfrak{J},\mathcal{K}) = (\mathfrak{M},\mathcal{Z}\times\mathcal{K})$ and $(\mathfrak{Q},\mathcal{C})\vee_*(\mathfrak{J},\mathcal{K}) = (\mathfrak{P},\mathcal{C}\times\mathcal{K})$, where for all $(z,\mathcal{k}) \in \mathcal{Z}\times\mathcal{K}$, $\mathfrak{M}(z,\mathcal{k}) = \mathfrak{S}'(\mathfrak{z}) \cup \mathfrak{J}'(\mathcal{k})$ and for all $(c,\mathcal{k}) \in \mathcal{C}\times\mathcal{K}$, $\mathfrak{P}(c,\mathcal{k}) = \mathfrak{Q}'(c) \cup \mathfrak{J}'(\mathcal{k})$. Assume that $(\mathfrak{M},\mathcal{Z}\times\mathcal{K}) \tilde{\cap} (\mathfrak{P},\mathcal{C}\times\mathcal{K}) = (\mathfrak{R},(\mathcal{Z}\times\mathcal{K}))$, where for all $(\mathfrak{z},\mathcal{k}) \in (\mathcal{Z}\times\mathcal{K})$,

$$\mathfrak{R}(\mathscr{k},\mathfrak{z})=\begin{cases}\mathfrak{M}(\mathfrak{z},\mathscr{k}), & (\mathscr{k},\mathfrak{z})\in(\mathcal{Z}\times\mathcal{K})-(\mathcal{C}\times\mathcal{K})=(\mathcal{Z}-\mathcal{C})\times\mathcal{K},\\ \mathfrak{M}(\mathfrak{z},\mathscr{k})\cap\mathfrak{P}(\mathfrak{z},\mathscr{k}), & (\mathscr{k},\mathfrak{z})\in(\mathcal{Z}\times\mathcal{K})\cap(\mathcal{C}\times\mathcal{K})=(\mathcal{Z}\cap\mathcal{C})\times\mathcal{K}.\end{cases}$$

Thus

$$\mathfrak{R}(\mathscr{k},\mathfrak{z})=\begin{cases}\mathfrak{S}'(\mathfrak{z})\cup\mathfrak{J}'(\mathscr{k}), & (\mathscr{k},\mathfrak{z})\in(\mathcal{Z}\times\mathcal{K})-(\mathcal{C}\times\mathcal{K})=(\mathcal{Z}-\mathcal{C})\times\mathcal{K}\\ [\mathfrak{S}'(\mathfrak{z})\cup\mathfrak{J}'(\mathscr{k})]\cap[\mathfrak{Q}'(\mathfrak{z})\cup\mathfrak{J}'(\mathscr{k})], & (\mathscr{k},\mathfrak{z})\in(\mathcal{Z}\times\mathcal{K})\cap(\mathcal{C}\times\mathcal{K})=(\mathcal{Z}\cap\mathcal{C})\times\mathcal{K}\end{cases}$$

Hence, $[(\mathfrak{S},\mathcal{Z})\,\tilde{\cup}\,(\mathfrak{Q},\mathcal{C})]\vee_*(\mathfrak{J},\mathcal{K})=_{\mathrm{M}}[(\mathfrak{S},\mathcal{Z})\vee_*(\mathfrak{J},\mathcal{K})]\,\tilde{\cap}\,[(\mathfrak{Q},\mathcal{C})\vee_*(\mathfrak{J},\mathcal{K})]$.

Proposition 21. Let $(\mathfrak{J},\mathcal{K}),(\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{Q},\mathcal{C})$ be SSs over U. Then,

I. $(\mathfrak{J},\mathcal{K})\vee_*[(\mathfrak{S},\mathcal{Z})\wedge(\mathfrak{Q},\mathcal{C})]\,\tilde{\subseteq}_{\mathrm{L}}\,[(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})]\vee[(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{Q},\mathcal{C})]$,

II. $(\mathfrak{J},\mathcal{K})\vee_*[(\mathfrak{S},\mathcal{Z})\vee(\mathfrak{Q},\mathcal{C})]\,\tilde{\subseteq}_{\mathrm{L}}\,[(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})]\wedge[(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{Q},\mathcal{C})]$.

Proof: let $(\mathfrak{S},\mathcal{Z})\wedge(\mathfrak{Q},\mathcal{C})=(\mathfrak{E},\mathcal{Z}\times\mathcal{C})$, where for all $(z,c)\in\mathcal{Z}\times\mathcal{C}$, $\mathfrak{E}(z,c)=\mathfrak{S}(z)\cap\mathfrak{Q}(c)$. Let $(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{E},\mathcal{Z}\times\mathcal{C})=(\mathfrak{R},\mathcal{K}\times(\mathcal{Z}\times\mathcal{C}))$, where for all $(\mathscr{k},(z,c))\in\mathcal{K}\times(\mathcal{Z}\times\mathcal{C})$,

$$\mathfrak{R}(\mathscr{k},(z,c))=\mathfrak{J}'(\mathscr{k})\cup[\mathfrak{S}(z)\cap\mathfrak{Q}(c)]'=\mathfrak{J}'(\mathscr{k})\cup[\mathfrak{S}'(z)\cup\mathfrak{Q}'(c)]$$

Assume that $(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})=(\mathcal{H},\mathcal{K}\times\mathcal{Z})$ and $(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{Q},\mathcal{C})=(\mathcal{M},\mathcal{K}\times\mathcal{C})$, where for all $(\mathscr{k},z)\in\mathcal{K}\times\mathcal{Z}$, $\mathcal{H}(\mathscr{k},z)=\mathfrak{J}'(\mathscr{k})\cup\mathfrak{S}'(z)$ and for all $(\mathscr{k},c)\in\mathcal{K}\times\mathcal{C}$, $\mathcal{M}(\mathscr{k},c)=\mathfrak{J}'(\mathscr{k})\cup\mathfrak{Q}'(c)$. Let $(\mathcal{H},\mathcal{K}\times\mathcal{Z})\vee(\mathcal{M},\mathcal{K}\times\mathcal{C})=(\beta,(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C}))$, where for all $((\mathscr{k},z),(\mathscr{k},c))\in(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C})$, $\beta((\mathscr{k},z),(\mathscr{k},c))=[\mathfrak{J}'(\mathscr{k})\cup\mathfrak{S}'(z)]\cup[\mathfrak{J}'(\mathscr{k})\cup\mathfrak{Q}'(c)]$.

Hence, for all $(\mathscr{k},(z,c))\in\mathcal{K}\times(\mathcal{Z}\times\mathcal{C})$, there exists $((\mathscr{k},z),(\mathscr{k},c))\in(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C})$ such that

$$\mathfrak{R}(\mathscr{k},(z,c))=\mathfrak{J}'(\mathscr{k})\cup[\mathfrak{S}'(z)\cup\mathfrak{Q}'(c)]=[\mathfrak{J}'(\mathscr{k})\cup\mathfrak{S}'(z)]\cup[\mathfrak{J}'(\mathscr{k})\cup\mathfrak{Q}'(c)]=\beta((\mathscr{k},z),(\mathscr{k},c)).$$

This completes the proof. It is obvious that the L-subset in Proposition 4.6. can not be L-equality with the following example:

Example 2. Let $\mathrm{E}=\{\ell_1,\ell_2,\ell_3,\ell_4\}$ be the PS, $\mathcal{K}=\{\ell_2,\ell_3\}$, $\mathcal{Z}=\{\ell_1\}$, and $\mathcal{C}=\{\ell_4\}$, be the subsets of E, $\mathrm{U}=\{\not{f}_1,\not{f}_2,\not{f}_3,\not{f}_4,\not{f}_5\}$ be the universal set, $(\mathfrak{J},\mathcal{K}),(\mathfrak{S},\mathcal{Z})$, and $(\mathfrak{Q},\mathcal{C})$ ve SSs over U such that $(\mathfrak{J},\mathcal{K})=\{(\ell_2,\{\not{f}_3,\not{f}_4\}),(\ell_3,\{\not{f}_2,\not{f}_3\})\}$, $(\mathfrak{S},\mathcal{Z})=\{(\ell_1,\mathrm{U})\}$ and $(\mathfrak{Q},\mathcal{C})=\{(\ell_4,\{\not{f}_2,\not{f}_3,\not{f}_4\})\}$.

We show that $(\mathfrak{J},\mathcal{K})\vee_*[(\mathfrak{S},\mathcal{Z})\wedge(\mathfrak{Q},\mathcal{C})]\neq_{\mathrm{L}}[(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})]\vee[(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{Q},\mathcal{C})]$.

Let $(\mathfrak{S},\mathcal{Z})\wedge(\mathfrak{Q},\mathcal{C})=(\mathfrak{E},\mathcal{Z}\times\mathcal{C})$, where $(\mathfrak{E},\mathcal{Z}\times\mathcal{C})=\{((\ell_1,\ell_4),\{\not{f}_2,\not{f}_3,\not{f}_4\})\}$. Assume that $(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{E},\mathcal{Z}\times\mathcal{C})=(\mathfrak{M},\mathcal{K}\times(\mathcal{Z}\times\mathcal{C}))$, where

$$(\mathfrak{M},\mathcal{K}\times(\mathcal{Z}\times\mathcal{C}))=\left(\left((\ell_2,(\ell_1,\ell_4)),\{\not{f}_1,\not{f}_2,\not{f}_5\}\right),\left((\ell_3,(\ell_1,\ell_4)),\{\not{f}_1,\not{f}_4,\not{f}_5\}\right)\right).$$

Let $(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{S},\mathcal{Z})=(\mathfrak{E},\mathcal{K}\times\mathcal{Z})$, where

$$(\mathfrak{E},\mathcal{K}\times\mathcal{Z})=\{((\ell_2,\ell_1),\{\not{f}_1,\not{f}_2,\not{f}_5\}),((\ell_3,\ell_1),\{\not{f}_1,\not{f}_4,\not{f}_5\})\}.$$

Suppose that $(\mathfrak{J},\mathcal{K})\vee_*(\mathfrak{Q},\mathcal{C})=(\mathfrak{R},\mathcal{K}\times\mathcal{C})$, where

$$(\mathfrak{R},\mathcal{K}\times\mathcal{C})=[((\ell_2,\ell_4),\{\not{f}_1,\not{f}_2,\not{f}_5\}),((\ell_3,\ell_4),\{\not{f}_1,\not{f}_4,\not{f}_5\})].$$

Let $(\mathfrak{E},\mathcal{K}\times\mathcal{Z})\vee(\mathfrak{R},\mathcal{K}\times\mathcal{C})=(\beta,(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C}))$. Thus,

$$(\beta,(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C}))$$
$$=\left\{\begin{matrix}\left(((\ell_2,\ell_1),(\ell_2,\ell_4)),\{\not{f}_1,\not{f}_2,\not{f}_5\}\right),\left(((\ell_2,\ell_1),(\ell_3,\ell_4)),\{\not{f}_1,\not{f}_2,\not{f}_4,\not{f}_5\}\right),\\ \left(((\ell_3,\ell_1),(\ell_2,\ell_4)),\{\not{f}_1,\not{f}_2,\not{f}_4,\not{f}_5\}\right),\left(((\ell_3,\ell_1),(\ell_3,\ell_4)),\{\not{f}_1,\not{f}_4,\not{f}_5\}\right)\end{matrix}\right\}.$$

Hence, $\beta((\ell_2,\ell_1),(\ell_3,\ell_4)) \neq \mathfrak{M}(\ell_2,(\ell_1,\ell_4))$, $\beta((\ell_2,\ell_1),(\ell_3,\ell_4)) \neq \mathfrak{M}(\ell_3,(\ell_1,\ell_4))$, $\beta((\ell_3,\ell_1),(\ell_2,\ell_4)) \neq \mathfrak{M}(\ell_2,(\ell_1,\ell_4))$, $\beta((\ell_3,\ell_1),(\ell_2,\ell_4)) \neq \mathfrak{M}(\ell_3,(\ell_1,\ell_4))$, implying that $(\beta,(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C})) \tilde{\nsubseteq}_L (\mathfrak{M},\mathcal{K}\times(\mathcal{Z}\times\mathcal{C}))$, and so $(\beta,(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C})) \neq_L (\mathfrak{M},\mathcal{K}\times(\mathcal{Z}\times\mathcal{C}))$.

5 | Int-Uni Decision-Making Method Applied to Soft Star-Product

This section applies the i*nt-uni* decision-making technique by applying the *int-uni* operator and int-uni decision function described by Çağman and Enginoğlu [11] to the soft star-product. Throughout this section, all the soft star-products (V_*) of the SSs over U are assumed to be contained in the set $V_*(U)$, and the approximation function of the soft star-product of $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$, that is $(\mathfrak{J},\mathcal{K})V_*(\mathfrak{S},\mathcal{Z})$

$\mathfrak{J}_{\mathcal{K}}V_*\mathfrak{S}_{\mathcal{Z}}: \mathcal{K}x\mathcal{Z} \rightarrow P(U)$,

where $(\mathfrak{J}_{\mathcal{K}}V_*\mathfrak{S}_{\mathcal{Z}})(k,z) = \mathfrak{J}'(k) \cup \mathfrak{S}'(z)$ for all $(k,z) \in \mathcal{K}x\mathcal{Z}$.

Definition 15. Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SS over U. Then, int-uni operators for soft star-product, denoted by $int_x uni_y$ and $int_y uni_x$ are defined respectively as

$int_x uni_y: V_* \rightarrow P(U)$, $int_x uni_y: (\mathfrak{J}_{\mathcal{K}}V_*\mathfrak{S}_{\mathcal{Z}}) = \cap_{k\in\mathcal{K}}(\cup_{z\in\mathcal{Z}}(\mathfrak{J}_{\mathcal{K}}V_*\mathfrak{S}_{\mathcal{Z}})(k,z)))$,

$int_y uni_x: V_* \rightarrow P(U)$, $int_y uni_x (\mathfrak{J}_{\mathcal{K}}V_*\mathfrak{S}_{\mathcal{Z}}) = \cap_{z\in\mathcal{Z}}(\cup_{k\in\mathcal{K}}(\mathfrak{J}_{\mathcal{K}}V_*\mathfrak{S}_{\mathcal{Z}})(k,z)))$.

Definition 16 ([11]). Let $(\mathfrak{J},\mathcal{K})V_*(\mathfrak{S},\mathcal{Z}) \in V_*(U)$. Then, the *int-uni* decision function for soft star-product, denoted by int-uni are defined by

int-uni: $V_* \rightarrow P(U)$, int-uni $(\mathfrak{J}_{\mathcal{K}}V_*\mathfrak{S}_{\mathcal{Z}}) = int_x uni_y (\mathfrak{J}_{\mathcal{K}}V_*\mathfrak{S}_{\mathcal{Z}}) \cup int_y uni_x(\mathfrak{J}_{\mathcal{K}}V_*\mathfrak{S}_{\mathcal{Z}})$.

The values *int-uni*$(\mathfrak{J}_{\mathcal{K}}V_*\mathfrak{S}_{\mathcal{Z}})$ is a subset of U called int-uni decision set of $\mathfrak{J}_{\mathcal{K}}V_*\mathfrak{S}_{\mathcal{Z}}$.

Given a set of parameters and a set of options, the int-uni decision-making method can be applied as follows to select an optimal collection of options while maintaining focus on the problem at hand:

Step 1. Identify workable subsets from the parameter set.

Step 2. Construct the SSs for each parameter set.

Step 3. Compute the soft star-product for the SSs.

Step 4. Generate the resulting int-uni decision set.

This approach allows for the application of SS theory to the int-uni decision-making problem, specifically in the context of soft star-product, demonstrating its usefulness in addressing decision-making scenarios.

Example 3. Mr. Ufuk and Ms. Duru began searching for a hotel for their honeymoon and went to meet with a travel agency regarding the matter. Since the travel agency offered too many options, the couple became confused. During this process, they decided to ELIMINATE hotels by first listing the parameters they definitely DO NOT want to have in the hotel they are planning to stay in. The set of hotels that the travel agency representative presented is represented by:

$U = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$.

The set of parameters used to determine the unpreferred hotels to ELIMINATE is: $E = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$, where:

I. a_1 = Expensive.

II. a_2= Far from the sea.

III. a_3= The absence of a water park.

IV. a_4= The absence of a hairdresser.

V. a_5= Has limited social activities and events.

VI. a_6= Far from the city center.

VII. a_7= Has a cafeteria that is not open 24 hours.

VIII. a_8= The absence of a fitness center.

They will make their decision using the uni-int decision-making method of the soft star-product. Since there are two decision-makers, the SS for Ms. Duru will be determined first, followed by the SS for Mr. Ufuk. Once both sets are established, the uni-int decision-making method on the soft star-product will be applied. The results from both sets will then guide the couple's final decision democratically. This process ensures that the decision is balanced and takes into account the preferences of both Mr. Ufuk and Ms. Duru. If these steps were to be outlined as an algorithm, they would proceed as follows:

Step 1. Determining the sets of parameters

The decision-makers' parameter sets are defined as follows. Şaheste, the representative of the travel agency, asks each decision maker to select the parameters that represent the characteristics they absolutely DO NOT want in the hotel they choose. These sets are defined as follows:

I. For Ms. Duru ($\mathcal{K}$): $\mathcal{K} = \{a_1, a_3, a_6\}$, meaning Ms. Duru does not want expensive hotels, lack a water park, or are far from the city center.

II. For Mr Utku ($\mathcal{Z}$): $\mathcal{Z} = \{a_2, a_5, a_7\}$, meaning Mr. Utku does not want hotels that are far from the sea, lack sufficient social activities and events, or have a cafeteria that is not open 24 hours.

These parameters represent undesirable qualities that make a hotel unsuitable for selection.

Step 2. Constructing the SSs using the PSs defined in Step 1

I. First, Ms. Duru's SS is determined by evaluating the parameters she DOES NOT want to be present in the hotel she selects.

II. Next, Mr. Ufuk's SS is determined by evaluating the parameters he DOES NOT want in his preferred hotel.

These SS are as follows $(\mathfrak{I}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$, respectively:

$(\mathfrak{I}, \mathcal{K}) =$
$\{(a_1, \{r_2, r_3, r_4, r_6, r_{10}, r_{13}\}), (a_3, \{r_3, r_4, r_6, r_{10}, r_{11}, r_{17}, r_{18}, r_{21}\}), (a_6, \{r_1, r_3, r_4, r_{10}, r_{13}, r_{17}, r_{19}\})\}$,
$(\mathfrak{S}, \mathcal{Z}) = \{(a_2, \{r_1, r_3, r_4, r_5, r_{10}, r_{17}, r_{21}\})), (a_5, \{r_1, r_3, r_{10}, r_{13}, r_{19}, r_{21}\}), (a_7, \{r_3, r_6, r_{10}, r_{17}, r_{21})\}\}$.

$(\mathfrak{I}, \mathcal{K})$ is an SS representing the hotels to be eliminated due to undesirable parameters in $\mathcal{K}$ according to Ms. Duru, $(\mathfrak{S}, \mathcal{Z})$ is an SS representing the hotels to be eliminated due to undesirable parameters in $\mathcal{Z}$ according to Mr. Ufuk. Note that the couple aims to ELIMINATE the hotels, not select them, as the representative of the travel agency has presented too many options.

Step 3. Determine the $\vee_*$-product of SSs:

$\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{I}_{\mathcal{Z}} =$
$\{((a_1, a_2), \{r_1, r_2, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}),$

$((a_1, a_5), \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}),$

$((a_1, a_7), \{r_1, r_2, r_4, r_5, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}),$

$((a_3, a_2), \{r_1, r_2, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}\}),$

$((a_3, a_5), \{r_1, r_2, r_4, r_5, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}\}),$

$((a_3, a_7), \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}\}),$

$((a_6, a_2), \{r_2, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}, r_{21}\}),$

$((a_6, a_5), \{r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{20}, r_{21}\}),$

$((a_6, a_7), \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}, r_{21}\})\},$

Step 4. Determine the set of int-uni $(\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{S}_{Z})$:

$\mathrm{int}_{k} - \mathrm{uni}_{z}(\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{S}_{Z}) = \bigcap_{k \in \mathcal{K}} \left(\bigcup_{z \in Z} ((\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{S}_{Z})(k, z)) \right).$

We first determine $\bigcup_{z \in Z} ((\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{S}_{Z})(k, z))$:

$(\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{S}_{Z})(a_1, a_2) \cup (\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{S}_{Z})(a_1, a_5) \cup (\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{S}_{Z})(a_1, a_7)$

$= \{r_1, r_2, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$\cup \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$\cup \{r_1, r_2, r_4, r_5, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$= \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$(\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{S}_{Z})(a_3, a_2) \cup (\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{S}_{Z})(a_3, a_5) \cup (\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{S}_{Z})(a_3, a_7)$

$= \{r_1, r_2, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}\}$

$\cup \{r_1, r_2, r_4, r_5, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}\}$

$\cup \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}\}$

$= \{r_1, r_2, r_4, r_5, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}\}$

$(\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{S}_{Z})(a_6, a_2) \cup (\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{S}_{Z})(a_6, a_5) \cup (\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{S}_{Z})(a_6, a_7)$

$= \{r_2, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$\cup \{r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{20}, r_{21}\}$

$\cup \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$= \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$

Thus

$(\mathrm{int}_{k} - \mathrm{uni}_{z})(\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{I}_{Z}) = \bigcap_{k \in \mathcal{K}} \left(\bigcup_{z \in Z} (\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{S}_{Z})(k, z)) \right) =$

$\{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$\cap \{r_1, r_2, r_4, r_5, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}\}$

$\cap \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$= \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$

is obtained.

$(\mathrm{int}_{z} - \mathrm{uni}_{k})(\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{S}_{Z}) = \bigcap_{z \in Z} \left(\bigcup_{k \in \mathcal{K}} ((\mathfrak{I}_{\mathcal{K}} \vee_* \mathfrak{S}_{Z})(k, z)) \right).$

We first determine $\cup_{\Bbbk \in \mathcal{K}}\left((\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{J}\mathfrak{S}_{Z})(\Bbbk, z)\right)$:

$(\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z})(a_1, a_2) \cup (\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z})(a_3, a_2) \cup (\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z})(a_6, a_2)$
$= \{r_1, r_2, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$\cup \{r_1, r_2, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}\}$

$\cup \{r_2, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$= \{r_1, r_2, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}.$

$(\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z})(a_1, a_2) \cup (\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z})(a_3, a_2) \cup (\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z})(a_6, a_2)$
$= \{r_1, r_2, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$\cup \{r_1, r_2, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}\}$

$\cup \{r_2, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$= \{r_1, r_2, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}.$

$(\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z})(a_1, a_5) \cup (\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z})(a_3, a_5) \cup (\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z})(a_6, a_5)$

$= \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$\cup \{r_1, r_2, r_4, r_5, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}\}$

$\cup \{r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{20}, r_{21}\}$

$\cup = \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}.$

$(\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z})(a_1, a_7) \cup (\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z})(a_3, a_7) \cup (\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z})(a_6, a_7)$

$= \{r_1, r_2, r_4, r_5, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$\cup \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}\}$

$\cup \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$= \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$

is obtained. Therefore,

$(\mathrm{int}_z - \mathrm{uni}_{\Bbbk})(\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z}) = \cap_{z \in Z}\left(\cup_{\Bbbk \in \mathcal{K}}\left((\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z})(\Bbbk, z)\right)\right) =$

$\cap \{r_1, r_2, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$\cap \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$\cap \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$= \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}, r_{21}\}.$

Thus

int-uni$(\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z}) = [\mathrm{int}_{\Bbbk} - \mathrm{uni}_z(\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z})] \cup [\mathrm{int}_z - \mathrm{uni}_{\Bbbk}(\mathfrak{J}_{\mathcal{K}} \mathrm{V}_{*} \mathfrak{S}_{Z})]$

$= \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$\cup \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}$

$= \{r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\}.$

In the hotel selection process, out of the 21 hotels, 19 were eliminated in the first stage. The remaining hotels, $\{r_3, r_{10}\}$ represent the options that meet the preferences of both Ms. Duru and Mr. Ufuk. The result will highlight the hotels that align with the preferences of all decision-makers involved. This process ensures a

balanced and democratic decision by considering both Ms. Duru's and Mr. Ufuk's preferences in selecting the best hotel.

6 | Conclusion

In this work, we introduced a new type of soft product, termed the soft star-product, based on Molodtsov's soft set theory. We provided an example and conducted a thorough analysis of its algebraic properties with respect to various types of soft subsets and equalities, including M-subset/equality, F-subset/equality, L-subset/equality, and J-subset/equality. We also examined the distributional properties of the soft star-product over different types of soft set operations. Finally, we applied the soft decision-making approach to soft star-product that simplifies the process by eliminating the need for rough or fuzzy soft sets, enabling the selection of optimal components from available options. We gave an illustrative example to show its effective application in diverse fields. This study paves the way for numerous applications, such as novel soft set-based cryptographic methods and new decision-making techniques. To further enrich the soft set literature both theoretically and practically, future research could propose additional soft product operations and explore fundamental properties related to various soft equal relations.

Conflict of Interest

The authors declare no conflict of interest.

Data Availability

All data are included in the text.

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